Heterogeneous Firms, Productivity and Poverty Traps*

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June, 2006

Abstract

We present a model of endogenous total factor productivity which generates poverty traps. We obtain multiple steady state equilibria for an arbitrarily small degree of increasing returns to scale. While the most productive firms operate across all the steady states, in a poverty trap less productive firms operate as well. This results in lower average firm productivity and lower total factor productivity. Our model is consistent with cross-country empirical evidence on differences in productivity and employment distribution across firms. In our model a growth miracle is accompanied by a shift of employment from small to large firms, consistent with the Industrial Revolution and Japan's post-war growth experiences.

JEL: L16, O11, O33, O40

Keywords: endogenous productivity, multiple equilibria, poverty traps

*We would like to thank Daron Acemoglu, Costas Azariadis, Larry Blume, Helge Braun, Paco Buera, Jim Bullard, Steve Durlauf, Nir Jaimovich, Per Krusell, Francesca Molinari, Karl Shell, Assaf Razin, Richard Rogerson, Gustavo Ventura, Ivan Werning, participants at Cornell-Penn State Macro Workshop, the IFPRI conference on "Threshold Effects and Non-Linearities in Growth and Development", and NBER Summer Institute for useful comments, discussions and constructive criticism. Any views expressed are our own and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System. All errors are our own. Corresponding author: Riccardo DiCecio, dicecio@stls.frb.org.

1 Introduction

We present an endogenous total factor productivity (TFP) model which leads to multiple steady state equilibria, and hence poverty traps. Our model is a variant of the neoclassical growth model with increasing returns introduced by Benhabib and Farmer (1994), with firms modelled in the tradition of Lucas (1978), Jovanovic (1982) and Hopenhayn (1992). There are many ex-ante identical potential firms which face an entry cost. Firms which choose to enter are entitled to produce an intermediate good with a productivity level drawn independently across firms from a given distribution. Because firms face a fixed operating cost, the decision to operate or not depends on the level of the firm's productivity. Productivity must be high enough so that the firm generates enough revenue (net of payments to factor inputs) to cover the operating cost. In other words, the operating cost defines a cutoff: firms with productivity above the cutoff choose to operate, the rest of the firms choose not to. The higher the cutoff, the more productive the average firm is.

The existence of multiple steady states depends on small demand externalities, which imply increasing returns to scale at the aggregate level. One of the main results of our paper is that poverty traps may occur for arbitrarily small values of increasing returns to scale.[1] Endogenizing TFP allows us to bridge the gap between poverty trap models based on increasing returns and the most recent empirical literature on the degree of returns to scale. An endogenous operating cost provides a powerful amplifying mechanism for increasing returns. We model the fixed operating cost as payments to overhead labor. Since the wage is endogenous, so is the lowest level of productivity used in the economy. This endogeneity may lead to multiple steady states. Consider an economy in a steady state with a high productivity cutoff and a large capital stock. The high cutoff implies that firms' average productivity is high. A large capital stock and high productivity imply that the wage is high, as is the operating cost. A high operating cost makes low productivity firms unprofitable, effectively cleansing the pool of firms. This justifies why the cutoff is high in the first place. Since only high productivity firms are operating, TFP is high. Conversely, in a steady state where capital is

[1] Galí (1995) obtains multiple equilibria and povery traps in a model where increasing returns stem from endogenous markups. However, the empirical evidence summarized in Section 3.4 below suggests that the degree of increasing returns is much smaller than the level required to obtain poverty trap in Galí's model.

low and lower productivity firms are operating (i.e. the cutoff is low), the wage is low. Since the wage is low, lower profits are sufficient to cover the operating cost. That is, the low operating cost sullies the pool of producers, leading to lower TFP and capital. Notice that in a good equilibrium high productivity firms produce more than in a bad equilibrium, despite facing a higher wage and the same interest rate. This is optimal because they face a higher demand for their goods, which offsets the contractionary pressure of higher factor prices.

An empirical motivation for our work comes from the studies of the determinants of cross-country income differences of Klenow and Rodriguez-Clare (1997), Hall and Jones (1999) and Caselli (2005). These authors find that income differences can be attributed, in part at least, to differences in TFP. Previous studies of poverty trap models with endogenous TFP pointed to the failure of adopting the most productive technology as the cause of low TFP and income in poor countries.[2] However, there is evidence pointing to the fact that differences in TFP across economies are related to the lowest level of firms' productivity. For example, Mokyr (1990, 2001) argues that the Industrial Revolution was characterized by a shift from less productive forms of production (workshops) to more productive ones (factories). Banerjee and Duflo (2005) cite the McKinsey Global Institute (2001) report on India, which finds that while larger production units (firms) use relatively new technologies, smaller (in home) production units have low productivity. Comin and Hobijn (2004) take a comprehensive look at the uses of various technologies as determinants of TFP and find that the key is not when new, better technologies are adopted, but when old, obsolete ones are let go of. Also, the empirical evidence on the importance of international knowledge spillovers summarized in Klenow and Rodriguez-Clare (2005) suggests that all countries can easily access frontier technologies.

A successful model of cross-country income and productivity differences should also provide a plausible story of how a "growth miracle" can occur, i.e. it should be consistent with the transition of a country from low to high output and productivity. In our model economy, a growth miracle is a transition from a bad equilibrium (low productivity cutoff) to a good one (high productivity cutoff). Such a take-off can be triggered by technological

[2]See, for example, Murphy, Shleifer, and Vishny (1989) and Ciccone and Matsuyama (1996). For comprehensive reviews of the literature on poverty traps see Matsuyama (2005) and Azariadis and Stachurski (2005).

progress which makes the highest productivity firms even more productive, or by a decline in the entry cost. In the first case, the increase in productivity of the best firms makes them more competitive, raising factor prices and driving low productivity firms out of business. In the second case, a decline in the entry cost brings about more competition from entering firms, driving out of the market low productivity firms. In both cases, along the transition path, the economy's TFP, output, capital, and firms' average productivity (and size) rise. An increase in the average firm size, caused by a massive shift of employment from small to large establishments, is a defining feature of the Industrial Revolution. A similar increase is recorded in the case of Japan's growth miracle. Between 1957 and 1969, the employment share of Japan's smallest establishments declined from 41 to 31.5 percent.

The rest of the paper is organized as follows. Section 2 presents the model, and Section 3 studies its steady state and dynamics properties. Section 4 describes some extensions of our basic model. Section 5 provides an interpretation of growth miracles that arises naturally in the model. We conclude in Section 6.

2 The Model

Our model is a variant of the neoclassical growth model. The model departs from the standard framework by having a richer structure of the production side of the economy. Firms are heterogenous: each firm has monopoly power over the good it produces, and firms have different productivity levels. Two features of the production side of the economy are crucial for the results of the paper:

1. a sunk entry cost;
2. an operating cost: in addition to capital and labor used directly in production, firms pay for a fixed amount of overhead labor.

A part of the entry costs stems from satisfying different official regulatory requirements (see Djankov, La Porta, Lopez-de-Silanes, and Shleifer, 2002). In addition, in some countries, entry requires significant side payments to local officials.[3] Entry cost may also include expenses related to acquisition

[3] In the case of Peru, this is documented by De Soto (1989).

of firm-specific capital,[4] acquisition of appropriate technology,[5] and market research.

The operating cost typically refers to overhead labor, and expenses that are lumpy in nature, for example, renting a physical location. According to the findings of Domowitz, Hubbard, and Petersen (1988), in U.S. manufacturing plants, the overhead labor accounts for 31percent of total labor. Ramey (1991) suggests that overhead labor is about 20 percent. The preferred estimate of overhead inputs in Basu (1996) is 28 percent.

We also assume that the firms learn their productivity only after the sunk entry cost is paid. This assumption reflects very high uncertainty faced by entering firms. This is routinely found in the data and documented, for example, by Klette and Kortum (2004) as a stylized fact.

2.1 Households

There is a continuum of households. They supply a fixed amount of labor, consume, and invest. They also own all firms in the economy. The problem of the representative household is given by

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t),\ \beta \in (0,1)$$
$$\text{s.t. } C_t + I_t = r_t K_t + w_t + \Pi_t + T_t,$$
$$I_t = K_{t+1} - (1-\delta) K_t.$$

where C_t denotes consumption, I_t is investment, K_t denotes the total household capital, r_t is the rental rate on capital, and w_t is the wage.[6] Π_t is the firms' profits, and T_t is a lump-sum transfer from the government; β and $\delta \in (0,1)$ are the discount rate and the depreciation rate, respectively. We assume a constant elasticity of substitution utility function with elasticity $\sigma > 0$.

[4] Ramey and Shapiro (2001) show that in some instances the specificity of firm capital is so extreme, that the sale price of such capital after a firm has been dissolved is only a tiny fraction of the original cost of capital.

[5] See, for example, Atkeson and Kehoe (2005).

[6] We assume that the household inelastically supplies one unit of labor.

2.2 Firms

2.2.1 Final Good Producers

The final consumption good in this economy is produced by perfectly competitive firms, according to the following production function:

$$Y_t = \left[\int_0^{\mu_t} [y_t(i)]^{\frac{1}{\lambda}} \, di \right]^{\lambda},$$

where μ_t is the number of intermediate goods produced in the economy, λ is a constant which is greater than one, and $y_t(i)$ is the quantity of the intermediate good i. Let $p_t(i)$ be the price of the i^{th} intermediate good relative to the final good. Then, the maximization problem of the final good producer can be written as

$$\Pi_t^{FF} = \max \left[\int_0^{\mu_t} [y_t(i)]^{\frac{1}{\lambda}} \, di \right]^{\lambda} - \int_0^{\mu_t} p_t(i) y_t(i) di,$$

and the first order optimality condition implies that the demand function for the i^{th} intermediate good is given by:

$$p_t(i) = \left[\frac{y_t(i)}{Y_t} \right]^{-\frac{\lambda - 1}{\lambda}}.$$

2.2.2 Intermediate Goods Producers

A firm in the intermediate goods sector lives one period, and is profit maximizing. All firms are ex-ante identical. There is a sunk entry cost κ. Once the entry cost is paid, a firm gains an ability to produce an intermediate good. The firm has monopoly power over the good it produces. Next, the firm draws a productivity parameter $A(j)$, where j is drawn from an i.i.d. uniform distribution over [0,1]. The production function for the good j is given by

$$[A(j)]^{1-\gamma} \left[k(j)^{\alpha} n(j)^{1-\alpha} \right]^{\gamma}$$

where $k(j)$ and $n(j)$ denote capital and labor respectively. The productivity parameter differs among the firms. A firm with a higher index has a higher productivity parameter, i.e. $A(j) > A(i)$ for $j > i$. In addition, function $A(j)$

is assumed to be continuous, and $A(0) = 0$. The parameter $\gamma \in (0, \lambda)$ determines the degree of returns to scale in variable inputs[7], and the parameter $\alpha \in (0, 1)$. We assume that λ is not too big.[8]

If a firm decides to produce, it must incur an operating cost in terms of wages paid to ϕ units of overhead labor. Consider the decision of a firm born in time t with a draw j. If it decides to produce, its profits are

$$\begin{aligned} &\pi_t^P(j) = \max_{k_t(j), n_t(j)} p_t(j)\, y_t(j) - r_t k_t(j) - w_t\left[n_t(j) + \phi\right] \\ &\text{s.t. } y_t(j) = [A(j)]^{1-\gamma}\left[k_t^{\alpha}(j) n_t^{1-\alpha}(j)\right]^{\gamma},\ p_t(j) = \left[\tfrac{y_t(j)}{Y_t}\right]^{\frac{1-\lambda}{\lambda}} \end{aligned} \tag{1}$$

where r_t denotes the rental rate on capital. Note that $r_t = R_t - (1 - \delta)$, where δ is the depreciation rate of capital used in production. The decision to produce or not depends on whether $\pi_t^P(j)$ is positive. Therefore, the j^{th} firm's profits $\pi_t^F(j)$ are given by:

$$\pi_t^F(j) = \max\{\pi_t^P(j), 0\}.$$

Free entry implies that, in equilibrium, firms' expected profits must be equal to the entry cost κ:

$$\int_0^1 \pi_t^F(j)dj = \kappa. \tag{2}$$

2.2.3 Firms' average productivity

We derive the equilibrium relationship between the firms' average productivity and the operating cost. First, we determine the lowest productivity level necessary for a firm to decide to produce. The existence of economy-wide competitive factor markets implies that in equilibrium, the gross profits, capital, and labor ratios of any two firms are equal to their (scaled) productivity ratio:

$$\frac{p_t(j)y_t(j)}{p_t(i)y_t(i)} = \frac{k_t(j)}{k_t(i)} = \frac{n_t(j)}{n_t(i)} = \frac{a(j)}{a(i)},\ \forall i, j \tag{3}$$

where $a(j) \equiv A(j)^{\frac{1-\gamma}{\lambda-\gamma}}$. The first order conditions of problem (1) imply that profits from producing are equal to the firm's share of the gross profits $(1 - \frac{\gamma}{\lambda})$

[7] This is what Lucas (1978) calls managers' span of control.

[8] In particular, $\lambda < 1 + \gamma \cdot \min((1-\alpha), \alpha)$. This assumption is not restictive at all, because for reasonable values of α and γ, it implies that λ should not be bigger than 1.30.

minus the operating cost:[9]

$$\pi_t^P(j) = (1 - \frac{\gamma}{\lambda})p_t(j)y_t(j) - \phi w_t.$$

Clearly $\pi_t^P(j)$ is increasing in j and, since $a(j) = 0$, there exists a cutoff firm, J_t, which is indifferent between producing or not:

$$(1 - \frac{\gamma}{\lambda})p_t(J_t)y_t(J_t) = \phi w_t. \tag{4}$$

Firms with indices higher than J_t will produce, and those with lower indices will not. Thus, firms' zero profit condition in (2) can be written as:

$$\kappa = \phi w_t \int_{J_t}^{1} \left[\frac{a(j)}{a(J_t)} - 1 \right] dj. \tag{5}$$

The previous equation defines the cutoff J_t as a function of the operating cost ϕw_t. An increase in the cutoff J_t has two effects: profits of every firm decline, and the number of producing firms as a fraction of entering firms declines. Therefore, the right hand side of (5) is decreasing in J_t, while it is clearly increasing in the fixed cost (ϕw_t). Hence, the cutoff is increasing in the operating cost. Therefore, firms' average productivity, $\bar{a}(J_t) = \frac{\int_{J_t}^1 a(j)dj}{1-J_t}$, is an increasing function of the operating cost.

2.2.4 Entry and the number of operating firms

Entry in this model refers to the number of firms which pay the entry cost κ. The number of entering firms differs from the number of operating firms because only a fraction of entrants will have productivity high enough to operate: the pool of producers consists only of firms which have an index higher than J_t. In particular, let ν_t denote the number of entering firms, and μ_t the number of operating firms. Then

$$\mu_t = \nu_t \int_{J_t}^{1} dj.$$

[9]Later on, with some abuse of terminology, we will refer to $(1 - \frac{\gamma}{\lambda})p_t(j)y_t(j)$ as firms gross profits.

2.3 Aggregate Output and TFP

Let K_t and N_t denote the total amount of capital and labor used by the firms:

$$K_t = \nu_t \int_{J_t}^{1} k_t(j)dj,$$

$$N_t = \nu_t \int_{J_t}^{1} [n_t(j) + \phi]\, dj = u_t N_t + \nu_t (1 - J_t)\, \phi,$$

where u_t is the fraction of labor used in production. Aggregate output can be written as

$$Y_t = \left[(\mu_t \bar{a}(J_t))^{(\lambda-\gamma)} u_t^{(1-\alpha)\gamma} \right] K_t^{\alpha\gamma} (N_t)^{(1-\alpha)\gamma}. \tag{6}$$

Finally, the rental rate on capital, the wage and the equation determining the cutoff J_t can be written as:

$$\alpha \frac{\gamma}{\lambda} \frac{Y_t}{K_t} = r_t, \tag{7}$$

$$(1-\alpha)\frac{\gamma}{\lambda} \frac{Y_t}{u_t N_t} = w_t, \tag{8}$$

$$(1 - \frac{\gamma}{\lambda}) \frac{a(J_t)}{\bar{a}(J_t)} \frac{Y_t}{(1-u_t)N_t} = \phi w_t. \tag{9}$$

2.4 Closing the Model

The resource constraint is given by:

$$C_t + K_{t+1} = Y_t + (1-\delta)K_t.$$

The only role the government has in the model is to collect the entry fees $\nu_t \kappa$ from firms and rebate them lump-sum to the households:

$$T_t = \nu_t \kappa.$$

Profits and the labor market clearing condition are:

$$\Pi_t = \Pi_t^{FF},$$
$$N_t = 1.$$

The definition of equilibrium is standard.

3 Properties of the Model

In this section we present some properties of the model economy developed in the previous section. In particular, we focus on the existence and the stability of the steady states. The main finding is that there can be multiple stable steady states with dramatically different levels of firms' average productivity, TFP, capital and output.

Intuitively, if there are multiple steady states, their existence is due to the endogenous productivity mechanism embedded in the model. To see this it is useful to start with a closer look at the key equation which determines the cutoff J, the zero profit condition in (5):

$$\kappa = \phi w_t \int_{J_t}^{1} \left[\frac{a(j)}{a(J_t)} - 1 \right] dj. \tag{10}$$

The equation above determines the relation between the cutoff J and the operating cost, ϕw_t. In particular, recall that the integral on the right hand side of this equation is decreasing in J. Thus, a higher operating cost translates into a higher cutoff, and vice-versa. In an economy where the operating cost is high, higher (gross) profits are required to cover this cost. Only high productivity firms can generate such profits. Therefore, the lower productivity firms are forced out from the pool of producers. This can be restated in broader terms: as the operating cost increases, the entry cost relative to operating cost falls, allowing more firms to enter. However, out of these firms, only the ones with higher productivity are profitable enough to operate. This relation between the operating cost and the cutoff provides economic intuition for the existence of multiple steady states. If multiple steady states exist, then one steady state will have high capital and only high productivity firms will be operating. High capital stock and high productivity imply that wage will be high, and so will be the operating cost. High operating cost, in turn, justifies why only high productivity firms will be operating. Finally, since productivity is high, a high capital stock is necessary to equate the return on capital to $1/\beta$. Conversely, in a "low" steady state, the capital stock and firms' average productivity will be low, and so will be the operating cost, allowing lower productivity firms to operate. Since firms' average productivity is low, the capital stock must be low to have the return on capital equal to $1/\beta$. A firm productive enough to be active in different steady states produces more in a good steady state than in a bad one, despite a higher wage and the same interest rate. This is optimal because it faces a higher demand for its goods, which offsets the contractionary pressure of higher factor prices.

3.1 Steady States

We present the argument formally in propositions 1 and 2, and provide proofs in appendix A. First, note that the number of firms is proportional to the total amount of labor used to cover the fixed cost:

$$\mu_t = \frac{1-u_t}{\phi} N_t.$$

Therefore, aggregate output is given by:

$$Y_t = TFP_t K_t^{\lambda s_k}, \tag{11}$$

where $s_k = \frac{\alpha\gamma}{\lambda}$ denotes the capital share of output, and total factor productivity is

$$TFP_t = \left[\phi^{\gamma-\lambda} \left[\bar{a}\left(J_t\right)\right]^{\lambda-\gamma} \left(1-u_t\right)^{\lambda-\gamma} u_t^{(1-\alpha)\gamma}\right]. \tag{12}$$

There are two components of TFP: firms' average productivity $\left(\bar{a}\left(J_t\right)\right)^{(\lambda-\gamma)}$, and the term $u_t^{(1-\alpha)\gamma}\left(1-u_t\right)^{\lambda-\gamma}$, which we call the *labor allocation* component. Firms' average productivity is increasing in J_t. The labor allocation component is a function of J_t as well, though not necessarily monotone. However, the effect of J_t on average productivity dominates, and TFP_t is increasing in J_t.

The following proposition allows us to present the model economy in a more familiar, neoclassical framework.

Proposition 1 *The aggregate production function in (6) and the total factor productivity in (12) are increasing in the cutoff J_t. The cutoff J_t, the wage w_t, and the aggregate output Y_t are all increasing functions of capital K_t. The rate of return on capital $R_t \equiv (r_t + 1 - \delta)$ is a function of K_t.*

Proof. See appendix A. ■

The proposition above implies that the dynamics of the economy can be characterized by the following system of difference equations:

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)^{1/\sigma} &= \beta R(K_{t+1}), \\ C_t + K_{t+1} &= Y(K_t) + (1-\delta)K_t, \end{aligned} \tag{13}$$

plus a transversality condition. We now turn to the existence and multiplicity of steady states.

Proposition 2 *The economy characterized by the system in (13) generically has an odd number of steady states. For any $\lambda > 1$, there exists a distribution of productivities, $a(j)$, such that the system (13) has multiple steady state equilibria.*

Proof. (sketch) Some straightforward manipulations of the first order conditions lead to the following relation between the rate of return on capital and the cutoff J :

$$r_t^{\frac{1-\alpha\gamma}{\alpha\gamma}} \kappa = \eta \cdot \Phi(J_t) \tag{14}$$

where

$$\Phi(J) \equiv \left[\frac{\bar{a}(J)}{\bar{a}(J) + \frac{\lambda-\gamma}{(1-\alpha)\gamma} a(J)} \right]^{\frac{\lambda-1}{1-\alpha\gamma}} a(J)^{\frac{\lambda-\gamma}{1-\alpha\gamma}} \int_J^1 \left[\frac{a(j)}{a(J)} - 1 \right] dj; \tag{15}$$

and η is a constant. Since $\Phi(J)$ is continuous and $\Phi(0) = \infty$, $\Phi(1) = 0$, there always exists a J^* which satisfies the equation below:

$$\left[1/\beta - (1-\delta)\right]^{\frac{1-\alpha\gamma}{\alpha\gamma}} \kappa = \eta \cdot \Phi(J^*). \tag{16}$$

In order for this equation to have more than one solution it is necessary that the function $\Phi(J)$ be increasing at some point (see Figure 2). In Appendix A we show that there always exists a function $a(j)$ such that this is the case. Note that (14) implies that if $\Phi(J)$ is increasing, so is $r(K)$. That is, the necessary condition for the existence of multiple steady states is that for some values of K the return on capital must be increasing. ■

Given propositions 1 and 2 it is easy to establish that the good economy has higher capital stock, higher output, higher total factor productivity, and higher firms' average productivity.

3.2 Dynamics

The following proposition characterizes the behavior of the economy around the steady state(s).[10]

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[10] An analysis of the global dynamics of our model is beyond the scope of this paper, and we refer the reader to Galí (1995).

Proposition 3 *Steady states with an odd index are saddles. Steady states with an even index can be classified as follows:*

1. *source, if* $Y' - \delta > \frac{\sigma C R'}{R}$ *and* $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 > 4\frac{\sigma C R'}{R}$;
2. *unstable spiral, if* $Y' - \delta > \frac{\sigma C R'}{R}$ *and* $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 < 4\frac{\sigma C R'}{R}$;
3. *sink, if* $Y' - \delta < \frac{\sigma C R'}{R}$ *and* $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 > 4\frac{\sigma C R'}{R}$;
4. *stable spiral, if* $Y' - \delta < \frac{\sigma C R'}{R}$ *and* $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 < 4\frac{\sigma C R'}{R}$.

Proof. See appendix A. ■

For the parameter values we consider in the rest of the paper, we obtain three steady states, with the odd steady state unstable (cases 1 and 2 in proposition 3). In comparing output and TFP across steady states we will focus on the two stable steady states.

3.3 Productivity Distribution and the Upper Bound on TFP Differences

So far we have shown that for some functions $a(j)$ there will be multiple stable steady states. The key property of the function $a(j)$ that generates multiplicity of equilibria is that $\bar{a}_J$ strongly dominates a_J for some J.[11]

A type of function that has this property is one that is nearly constant on some interval (J_1, J_2). The larger this interval is, the farther apart the stable steady states are from each other. In terms of firms' productivity distribution, this translates into the lower steady state having a large number of firms with nearly the same low productivity.

The property of the function $a(j)$ established above yields a surprising result about the magnitude of the differences between the steady states. Recall that the limiting case of λ equal to one has the least favorable implications for the existence of multiple steady states, because the model essentially collapses to the standard neoclassical model. Thus, it is important to see how large the steady state differences can be for λ arbitrarily close to one. The condition for the existence of multiple steady states translates to $a(J)$ being

[11] See proof of propositon 2 in appendix A.

(almost) a constant over some interval. In this case, the extremes of this interval correspond to the two steady state values of J. Since $a(J)$ is constant, the differences between TFP are due only to differences in:

$$\left[\frac{\bar{a}(J)}{\bar{a}(J)+\frac{1-\gamma}{(1-\alpha)\gamma}a(J)}\right]^{1-\alpha\gamma}.$$

The upper and the lower bounds for this object are

$$LB\equiv\left[\frac{\gamma-\alpha\gamma}{1-\alpha\gamma}\right]^{1-\alpha\gamma}<\left[\frac{\bar{a}(J)}{\bar{a}(J)+\frac{1-\gamma}{(1-\alpha)\gamma}a(J)}\right]^{1-\alpha\gamma}<1\equiv UB.$$

The lower bound depends on γ, and the share of capital $s_k=(\alpha\gamma)/\lambda$. The corresponding upper bounds on the TFP differences are presented in tables 1 and 3.

3.4 Numerical Examples

The numerical examples below illustrate that for the range of parameters λ,γ and α typically used in the literature, the differences between the steady states can be very substantial. In particular, with conservative values of λ,γ and α, aggregate output and capital in the high steady state can be 40 percent higher than these quantities in the lower steady state.

There are seven parameters in the model: β, δ, ϕ, κ, λ, γ, and α, which must be chosen before solving the model. The model's implications are robust to the choice of β and δ for the commonly used values of $\beta\in(0.94,0.99)$ and $\delta\in(0.08,0.12)$. Therefore, we set $\beta=0.95$, and $\delta=0.10$. We normalize ϕ to one[12] and we consider different values for κ, chosen in line with the findings of Djankov, La Porta, Lopez-de-Silanes, and Shleifer (2002). The other three parameters, λ,γ, and α, deserve more consideration.

The first parameter, λ, governs the degree of increasing returns to scale in the economy. There has been a large debate in the recent literature on the magnitude of increasing returns in the economy. While earlier researchers (most notably, Hall (1988)) suggested that there are large increasing returns to scale in the economy, subsequent work has shown that the returns to

[12] Notice that for our results only $\kappa/(\phi^n)^{\frac{\lambda-\gamma}{1-\alpha\gamma}}$ matters (see equations 14, and 25 in appendix A).

scale can be best described as constant to moderately increasing. The latest estimates of λ are probably those constructed by Laitner and Stolyarov (2004). Their preferred point estimate is $\lambda = 1.10$, with confidence interval $(1.03, 1.20)$. These numbers are not far from the estimates of Bartelsman, Caballero, and Lyons (1994), Burnside (1996), Burnside, Eichenbaum, and Rebelo (1995), Basu (1996), Basu and Fernald (1997), and Harrison (2003). Because of the above, we restrict λ to be between 1 and 1.25.

The next parameter, γ, represents the share of output that goes to capital and labor used directly in production, for a given value of λ. As a benchmark, we consider $\gamma = 0.85\lambda$, which is the preferred value of Atkeson and Kehoe (2005). This is very close to the estimated value of 0.84 in Basu (1996). Other values of γ which we consider are 0.80λ, 0.90λ and 0.95λ.[13]

The choice of the next parameter, α, depends on the interpretation of s_k. Interpreted literally, this is the capital share of output. However, if a part of firms' (entrepreneurs') share of output, i.e. $(1 - \gamma/\lambda)$, is interpreted as capital income, then s_k is less than the capital share of output. With this interpretation, one needs to take a stand on how the firm's share of output is divided between capital and labor. A commonly used rule is to split this share so that the capital share of output is α. As a starting point, we set s_k to 0.36. This implies, for example, that when γ is set to 0.85λ, α is equal to 0.42.

Tables 1-4 present the resulting differences in values of output and TFP across the steady states. Note that the difference in the levels of TFP for $\lambda = 1.01$ is very close to the theoretical upper bounds constructed earlier. The differences in the levels of TFP translate into substantial differences in levels of capital and in levels of output. In particular, even for $\lambda = 1.01$, the economy in the good steady state produces 27 percent more output than the economy in the bad steady state.

In the studies of the long run behavior of an economy, using the proper measure of capital share of output is of crucial importance. For example, for

[13] Note that in the model there is a difference between aggregate returns to scale and firm level returns to scale. While at the aggregate level there are increasing returns to scale, at the firm level, as long as $\gamma < 1$, the returns to scale in *variable* inputs are decreasing. In our model, heterogenous productivity leads to a heterogenous degree of returns to scale in *all* inputs. For firms with higher productivity, the decreasing returns to scale in variable inputs dominate the increasing returns to scale effect of the fixed cost; for the firms with lower productivity, it is the opposite. These observations are broadly consistent with empirical findings of Basu (1996), and Basu and Fernald (1997).

the unified theory of Parente and Prescott (2005) to be successful, the capital share of output should be between 0.55 and 0.65. The magnitude of this share depends on the definition of investment (capital). In the context of this paper it is proper to define investment as "any allocation of resources that is designed to increase future productivity" (see Parente and Prescott, 2000). That is, investment should include maintenance and repair, research and development, software, investment in organizational capital, and investment in human capital. Parente and Prescott (2000) find that including these items in investment implies that the capital share of output is larger than 0.50 and can reach as high as 2/3.[14]

For the model developed above, the capital share is important for two reasons. First, there is the standard "neoclassical" effect: the higher the capital share is, the higher the effect of the TFP is on the economy. To see this note that for two identical economies, differing only in TFP, the steady state capital ratio relates to the TFP ratio as follows:

$$\frac{K^{HIGH}}{K^{LOW}} = \left[\frac{TFP^{HIGH}}{TFP^{LOW}}\right]^{\frac{1}{1-s_k \cdot \lambda}}.$$

Clearly, the higher the share of capital is, the higher the difference in steady state capital is between the two economies.

Second, the capital share directly impacts TFP, because it enters into the definition of TFP in (12), and into the definition of the function $\Phi(J)$ in (15). Because of the highly non-linear nature of TFP and Φ as functions of the cutoff J, it is not possible to analytically derive the effect of an increase in the capital share of output on resulting TFP differences across the steady states. However, when λ tends to one, it can be shown that the theoretical upper bound on these differences gets larger as the capital share grows. Indeed, recall that this upper bound is given by:

$$\frac{UB}{LB} = \left[\frac{1-s_k}{\gamma - s_k}\right]^{1-s_k}.$$

Since $\gamma < 1$, an increase in the capital share leads to an increase in UB/LB. For all numerical experiments (table 3) the increase in the capital share

[14] For details and references see the original paper. A large portion of the unmeasured capital is organization capital. Findings of Atkeson and Kehoe (2005) imply that the value of organizational capital in the US manufacturing sector is *larger* than the value of physical capital.

of output increases the TFP differences. Combined with the "neoclassical effect" described above, this leads to even larger differences in output and in capital across the steady states (table 4).

Differences across steady states become very large, as long as either λ or s_k is large. For example, for $s_k = 0.65$ and $\lambda = 1.01$, output differs across the steady states by a factor of 1.67, while with $s_k = 0.36$ and $\lambda = 1.25$ the economy in the good steady state produces 1.48 times more output than the economy in the bad steady state. When both s_k and λ are high, the resulting differences in output and in capital are huge, reaching as much as $4,600$ percent.

		$\boldsymbol{\lambda}$						
		1*	**1.01**	**1.05**	**1.1**	**1.15**	**1.2**	**1.25**
γ	$\mathbf{0.95\boldsymbol{\lambda}}$	1.04	1.04	1.05	1.05	1.05	1.06	1.06
	$\mathbf{0.9\boldsymbol{\lambda}}$	1.1	1.1	1.1	1.11	1.12	1.13	1.14
	$\mathbf{0.85\boldsymbol{\lambda}}$	1.17	1.16	1.17	1.19	1.21	1.22	1.24
	$\mathbf{0.8\boldsymbol{\lambda}}$	1.26	1.24	1.26	1.28	1.31	1.35	1.38

*Theoretical upper bound for $\lambda \to 1$.

Table 1: Relative TFP and returns to scale ($s_k = 0.36$)

		$\boldsymbol{\lambda}$					
		1.01	**1.05**	**1.1**	**1.15**	**1.2**	**1.25**
γ	$\mathbf{0.95\boldsymbol{\lambda}}$	1.07	1.08	1.09	1.1	1.11	1.12
	$\mathbf{0.9\boldsymbol{\lambda}}$	1.16	1.17	1.19	1.22	1.24	1.27
	$\mathbf{0.85\boldsymbol{\lambda}}$	1.27	1.3	1.33	1.38	1.42	1.48
	$\mathbf{0.8\boldsymbol{\lambda}}$	1.4	1.47	1.51	1.6	1.69	1.82

Table 2: Relative output and returns to scale ($s_k = 0.36$)

4 Extending the Basic Model

In this section we consider two possible extensions of our basic model. First, we consider the implications of endogenizing the entry cost. Then, we analyze infinitely lived firms.

		$\boldsymbol{\lambda}$						
		1*	**1.01**	**1.05**	**1.1**	**1.15**	**1.2**	**1.25**
$\mathbf{s}_k$	**0.36**	1.17	1.16	1.17	1.19	1.21	1.22	1.24
	0.45	1.18	1.17	1.18	1.2	1.22	1.24	1.27
	0.55	1.19	1.18	1.2	1.22	1.26	1.3	1.37
	0.6	1.19	1.19	1.21	1.24	1.3	1.38	1.55
	0.65	1.2	1.2	1.23	1.29	1.39	1.63	2.05

*Theoretical upper bound for $\lambda \to 1$.

Table 3: Relative TFP and capital share ($\gamma = 0.85$)

		$\boldsymbol{\lambda}$					
		1.01	**1.05**	**1.1**	**1.15**	**1.2**	**1.25**
$\mathbf{s}_k$	**0.36**	1.27	1.3	1.33	1.38	1.42	1.48
	0.45	1.33	1.37	1.43	1.51	1.61	1.74
	0.55	1.44	1.53	1.66	1.86	2.19	2.76
	0.6	1.53	1.66	1.9	2.32	3.2	5.76
	0.65	1.67	1.9	2.41	3.71	9.16	46.49

Table 4: Relative output and capital share ($\gamma = 0.85$)

4.1 Entry and Operating Costs

The key feature of the model that allows for multiple steady state equilibria is the asymmetry between the entry and the operating cost. While the operating cost is endogenous and changes with the state of the economy, the entry cost is not. One might try to relax this assumption, and allow both the entry and the operating costs to be endogenous. In this case, multiple steady state equilibria may exist as long as a weaker form of asymmetry is preserved. In particular, the operating cost should be "more" increasing in capital than the entry cost, so that the ratio of the operating cost to the entry cost is increasing in capital. We suggest a simple example, based on Atkeson and Kehoe (2005). Let the entry cost take a form of κ units of entry services which firms need to purchase to enter. Let the production function of these services be exactly the same as it is for consumption goods, except that it is more or less labor intensive. Then, it can be shown that in a steady state the zero profit condition in (10) becomes

$$\kappa = \eta w^{\omega} \int_J^1 \left[\frac{a(j)}{a(J)} - 1 \right] dj, \tag{17}$$

where η and ω are positive constants. When ν is one, it is the same zero profit condition as before. When ω is zero, it is the case of Atkeson and Kehoe (2005). As long as ω is not zero, the key relation between wage w and the cutoff J, which leads to multiple steady states, is preserved.

4.2 Infinitely Lived Firms

The firms' productivity changes over time. We consider two opposite cases:

1. firms' productivity in every period is given by $A(j)$, where j is the original draw.

2. firms draw a new j which is independent of past draws.

We also assume that firms are dying with a constant probability $1 - p$. Consider a period-t decision of a firm born in time s with a draw j. The Bellman equation of this firm is

$$V_t(s, j) = \max \left[\pi_t^P(s, j) + \frac{p}{R_{t+1}} E_t V_{t+1}(s + 1, j') \right] \tag{18}$$

where $\pi_t^P(s, j)$ is the profits from producing as defined in Section 2.2.2, and R_{t+1} is the rate of return on capital (i.e. the interest rate).[15] The law of motion of j is $j' = j$ in case 1, and j' i.i.d. uniform over [0,1] in case 2. The free entry condition implies that

$$\kappa = \int_0^1 V_t(t, j) dj \tag{19}$$

Proposition 2 extends to both cases. If firms' productivity is the same as the original draw, the function Φ is unchanged. For the case of i.i.d. draws, the function Φ is replaced by the following:

[15] Recall that the firms are owned by the households and there is no aggregate uncertainty in the economies we consider. Therefore, $1/R_{t+1}$ is the relevant discount factor.

$$\tilde{\Phi}(J) \equiv \frac{\left[\frac{\bar{a}(J)}{\bar{a}(J)+\frac{(\lambda-\gamma)}{(1-\alpha)\gamma}\left(1+\frac{p}{R}\int_J^1\left[\frac{a(j)}{a(J)}-1\right]dj\right)a(J)}\right]^{\frac{\lambda-1}{1-\alpha\gamma}}}{\left(1+\frac{p}{R}\int_J^1\left[\frac{a(j)}{a(J)}-1\right]dj\right)^{1-\frac{(\lambda-\gamma)}{1-\alpha\gamma}}} a(J)^{\frac{\lambda-\gamma}{1-\alpha\gamma}} \int_J^1 \left[\frac{a(j)}{a(J)}-1\right] dj \tag{20}$$

Notice that when $p = 0$, case 2 simplifies to our baseline model in Section 2, i.e. $\tilde{\Phi}(J)|_{p=0} = \Phi(J)$.

5 Growth Miracles: an Interpretation

A puzzle closely related to cross-country income differences is the question of how and why countries grow and what causes growth miracles. A common view in the literature is that growth miracles are a result of a dramatic shift towards more productive firms and better forms of industrial organization. For example, Mokyr (2001) states that the Industrial Revolution was accompanied by "*the ever-growing physical separation of the unit of consumption (household) from the unit of production (plant),...*" due to "*... concentration of former artisans and domestic workers under one roof (plants), in which workers were more or less continuing what they were doing before, only away from home ...*" and "*... a more radical change in production technique, with substantial investment in fixed capital combined with strict supervision and rigid discipline.*" Thus, plants and factories (i.e. bigger establishments) must have been more productive than "in home" production units (i.e. the smallest establishments), and the Industrial Revolution can be viewed as a shift of resources from the smallest, less productive units to larger, more productive ones.

An intriguing question is whether other growth miracles are similar in this respect to the Industrial Revolution. One way to shed light on this question is to ask what happens to the share of labor employed in the smallest establishments during such miracles. The data to answer this question is available for Japan and it reveals a striking pattern (see Figure 1): the labor share of the smallest establishments (i.e. establishments with nine employees or fewer) fell by 9 percent between 1957 and 1969. The period from 1957 to 1969 was a period of remarkable economic growth, which Parente and Prescott (2005) classify as a period of a growth miracle. Even more interestingly, in countries

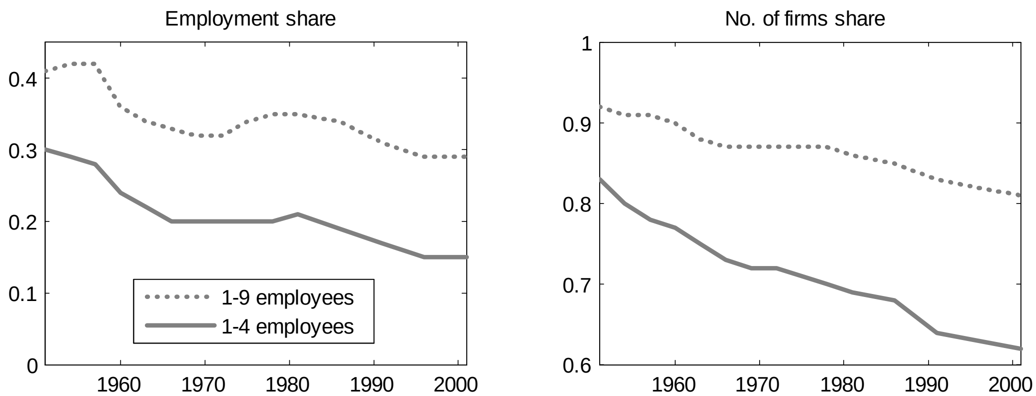

Figure 1: Smallest establishments in Japan: employment and number of firms shares.

which have yet to start catching up with developed countries, the smallest firms have the largest employment share.[16] In developed countries it is the opposite: the largest firms have the largest share of employment. Thus, a successful model of cross-country income differences should be able to generate growth miracles which are accompanied by a shift in employment from the smallest establishments to larger, more productive ones.

Such a shift in our model's framework depends on the properties of the function $a(j)$. If the corresponding probability density function of productivity is one which implies the existence of multiple steady states, i.e. it has a high density somewhere at the lower tail, then a shift from the smallest to the largest establishments occurs when the economy moves away from a "low J" steady state to a "high J" steady state.

There are two reasons that can cause such a shift.[17] The first one, is the decline in the entry barriers, i.e. the decline in the entry cost κ.

To illustrate this point, it is useful to start with Figure 2. An interesting pattern emerges. For larger values of κ, there is a unique, low-cutoff steady state, and for lower κ's there is a unique steady state, with large J. For

[16] See Table 1 in Tybout (2000, p.16).

[17] Even in the case of a unique steady state, a reduction in the entry cost or technological progress can change the location of the steady state dramatically. However, if the steady state is unique, cross-country differences in output and TFP can be attributed to differences in fundamentals.

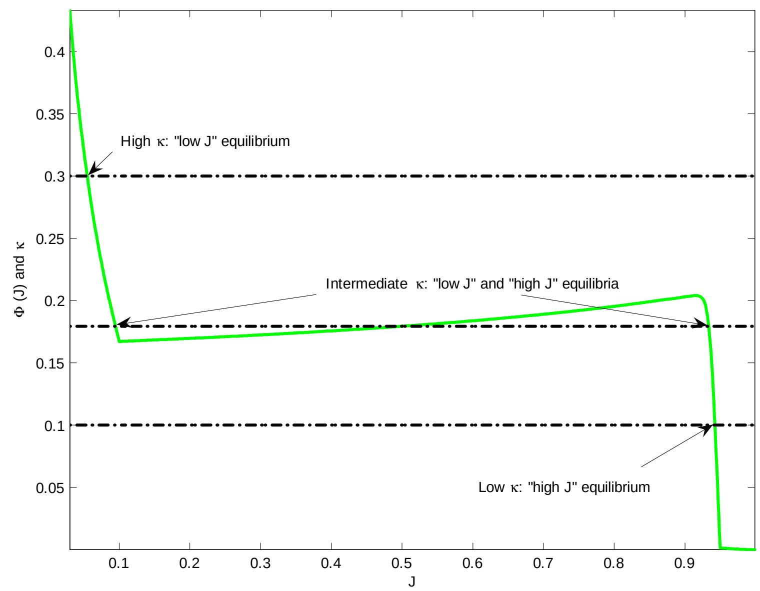

Figure 2: The role of the entry cost κ.

intermediate values of κ there can be two steady states. A small change in the value of κ can lead to large differences in J and the corresponding values of capital and output. In our model economy, the best technologies available are used regardless of the magnitude of the entry cost. The usage of worse technologies, on the other hand, depends on the entry cost. A reduction in the entry cost can cleanse the economy of lower productivity firms, increasing firms' average productivity and TFP. This mechanism of growth miracles shares a common driving force, reduction of barriers, with the one of Parente and Prescott (2000). However, the effect of the reduction of the barriers is different. In their model new, better technologies are not being used because of the barriers. Here, the entry barriers determine not the highest, but the lowest level of technology that is being used in the economy.

The second reason for a growth miracle is technological progress. A natural way to introduce this into our model is to consider a one-time permanent increase in the function $a(j)$ for values of j close to one.[18] That is, the best

[18] A better model to address the effect of productivity improvements would be one where the highest level of technology that is available in the economy grows over time. Building and examining such a model is left for future research.

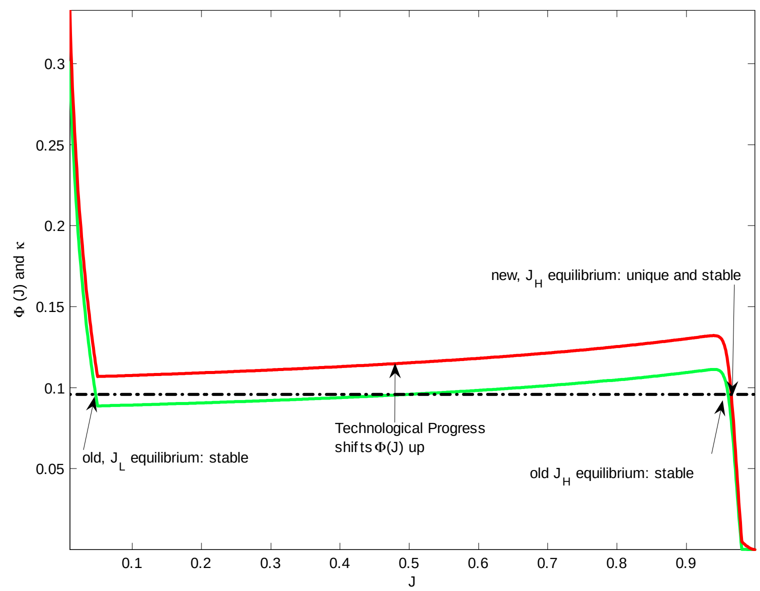

Figure 3: A growth miracle driven by technological progress.

technologies become even better. Mathematically, this can be written, for example, as

$$a(j) = \begin{cases} a(\bar{j}) \left[\frac{a(j)}{a(\bar{j})}\right]^q, \text{ if } j \geq \bar{j}, \\ a(j), \text{ otherwise;} \end{cases}$$

where $\bar{j}$ is close to one, and q is greater than one. For any $J < \bar{j}$, the change in the function $a(j)$ will cause $\Phi(J)$ to rise. If such a rise is sufficiently large, the "low J" steady state will disappear (see Figure 3), and the economy will start growing toward a "high J" steady state.

The resulting change in the distribution of employment across firms of different sizes is reported in the first column of Figure 4. The differences in the employment distribution between "high J" and "low J" countries are very similar to those between an average low income country and the U.S., in the sample of Tybout (2000).

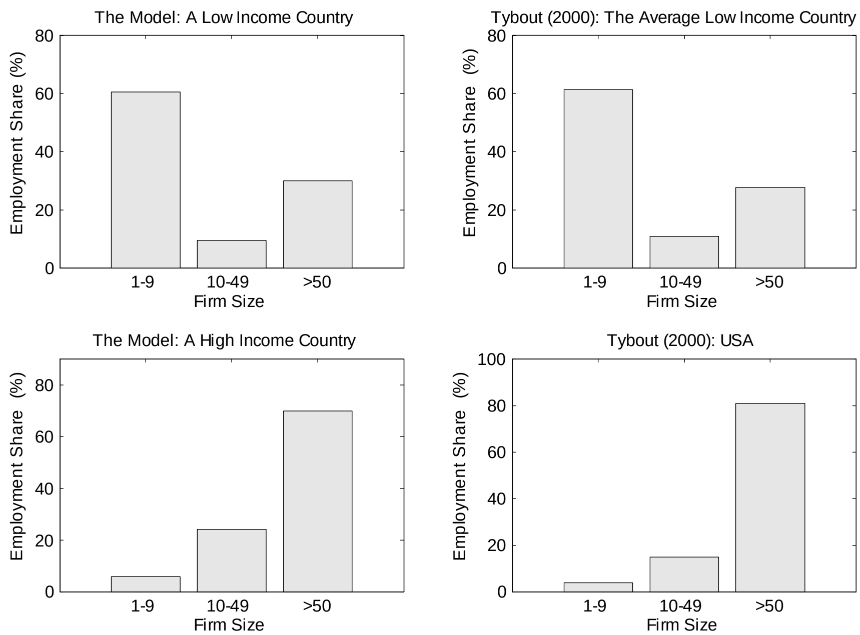

Figure 4: Employment Shares: model versus data.

6 Conclusions

Recent empirical studies attribute a sizable fraction of cross-country income differences to differences in TFP. These differences reflect, in part, the fact that the fraction of low productivity firms in less developed countries is much higher than in industrialized countries.

We introduce heterogeneity in productivity across firms in an otherwise standard model. In our model differences in TFP arise endogenously, and we obtain multiple steady state equilibria for an arbitrarily small degree of increasing returns to scale. Economies with the same fundamentals can be at very different steady states. If an economy is in a good steady state only the most productive firms operate, leading to high TFP, capital and output. In an economy locked in a poverty trap the pool of producers is sullied by low productivity firms, with low TFP, capital and output.

In our model a growth miracle, induced by technological progress or a decline in entry barriers, is accompanied by a shift of employment from small to large firms. This is consistent with the Industrial Revolution and Japan's post-war growth experiences.

Finally, our model's implications for the employment distribution across firms of different sizes is consistent with the empirical evidence.

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A Proofs of Propositions

A.1 Proof of Proposition 1

Equations in (8) and (9) imply that the fraction of labor used in production u_t is a function only of the cutoff J_t :

$$u_t = \frac{\bar{a}\left(J_t\right)}{\bar{a}\left(J_t\right) + \frac{\lambda-\gamma}{(1-\alpha)\gamma} a(J_t)}.$$

Substituting this expression of u_t into the equation (12), we get that

$$TFP\left(J_t\right) = \phi^{\gamma-\lambda}\left[\frac{\frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)}{\bar{a}\left(J_t\right) + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)}\right]^{\lambda-\gamma}$$
$$\left[\frac{\bar{a}\left(J_t\right)}{\bar{a}\left(J_t\right) + \frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)}\right]^{(1-\alpha)\gamma} \left[\bar{a}\left(J_t\right)\right]^{\lambda-\gamma} \tag{21}$$

Differentiating the previous expression:

$$\text{signum}\left(TFP_J\right) = \text{signum}\left[\begin{array}{c} \frac{(\lambda-\gamma)(1-\alpha)\gamma}{(1-\alpha)\gamma\bar{a}+(\lambda-\gamma)a}\frac{a_J}{a}\left(\bar{a}-a\right) + \\ \bar{a}_J\left(\frac{1}{\bar{a}} - \frac{1}{\bar{a}+\frac{\lambda-\gamma}{(1-\alpha)\gamma}a}\right)\end{array}\right], \tag{22}$$

where

$$TFP_J = \frac{\partial TFP\left(J\right)}{\partial J},\ a_J = \frac{\partial a\left(J\right)}{\partial J},\ \bar{a}_J = \frac{\partial \bar{a}\left(J\right)}{\partial J}.$$

The terms in parenthesis in (22) are positive and they are multiplied by positive terms. Hence, $TFP_J > 0$.

Using the firms' first order condition in (8) and the zero profit condition in (5) we get that the following relation between the cutoff J_t and capital K_t :

$$\begin{aligned}\kappa &= (1-\alpha)\frac{\gamma}{\lambda}\left[\frac{\lambda-\gamma}{(1-\alpha)\gamma}\right]^{\lambda-\gamma}\left[\frac{\bar{a}(J_t)}{\bar{a}(J_t)+\frac{\lambda-\gamma}{(1-\alpha)\gamma}a(J_t)}\right]^{(1-\alpha)\gamma+(\lambda-\gamma)-1} \\ &\quad \left[a(J_t)\right]^{\lambda-\gamma}K_t^{\alpha\gamma}\left[\frac{1}{a(J_t)}\int_{J_t}^{1}a(j)dj - (1-J_t)\right]\end{aligned}$$

For a given K_t the left hand side of this equation varies with J_t from $+\infty$, to zero. Moreover, one can easily show that the left hand side is decreasing in J_t. Thus, there exists a unique J_t which solves the equation. In addition, it is increasing in K_t. Because J_t is increasing in K_t, so is output Y_t, and wage w_t. In addition, since for a given K_t output Y_t is uniquely determined, so is the R_t, i.e. R_t is a function of K_t. ■

A.2 Proof of Proposition 2

Use equations (7) and (11) to express K_t as a function of r_t and J_t. Then, substituting this expression of K_t into the equation (21). Finally, using equation (8), we get

$$r_t^{\frac{1-\alpha\gamma}{\alpha\gamma}} \kappa = \eta \cdot \Phi(J_t) \tag{23}$$

where

$$\Phi(J) \equiv \left[\frac{\bar{a}(J)}{\bar{a}(J) + \frac{\lambda-\gamma}{(1-\alpha)\gamma} a(J)} \right]^{\frac{\lambda-1}{1-\alpha\gamma}} a(J)^{\frac{\lambda-\gamma}{1-\alpha\gamma}} \int_J^1 \left[\frac{a(j)}{a(J)} - 1 \right] dj, \tag{24}$$

and η is a constant:

$$\eta = \phi^{\frac{\gamma-\lambda}{1-\alpha\gamma}} \left(1 - \frac{\gamma}{\lambda}\right) \left(\frac{\alpha\gamma}{\lambda}\right)^{\frac{\alpha\gamma}{1-\alpha\gamma}} \left[\frac{\lambda-\gamma}{(1-\alpha)\gamma} \right]^{\frac{\lambda-\gamma}{1-\alpha\gamma}-1}. \tag{25}$$

Since $\Phi(J)$ is continuous and $\Phi(0) = \infty$, $\Phi(1) = 0$, there always exists a J^* which satisfies the equation below:

$$[1/\beta - (1-\delta)]^{\frac{1-\alpha\gamma}{\alpha\gamma}} \kappa = \eta \cdot \Phi(J^*). \tag{26}$$

We now have to show that for any J^* satisfying equation (26) there exists a pair (c^*, K^*), both positive, such that $R(K^*) = 1/\beta$, and $c^* = Y(K^*) - \delta K^*$. This is an immediate consequence of proposition 1.

If there is more than one J^* satisfying equation (26), then there will be multiple steady states. Note that for given parameters λ, γ, and α, the shape of the function $\Phi(J)$ is entirely determined by the shape of function $a(j)$: If $a(j)$ is such that $\Phi_J > 0$ then (26) has multiple solutions. To conclude the proof, we must show that there exists a function $a(j)$ such that $\Phi_J > 0$. The sign of Φ_J can be checked as follows:

$$\text{signum}\left(\Phi_J\right) = \text{signum}\left\{ \begin{array}{c} \frac{\lambda-1}{1-\alpha\gamma}\left[\frac{\bar{a}_J}{\bar{a}} - \frac{\bar{a}_J+\frac{\lambda-\gamma}{(1-\alpha)\gamma}a_J}{\bar{a}+\frac{\lambda-\gamma}{(1-\alpha)\gamma}a}\right] \\ +\left[\frac{\lambda-\gamma}{1-\alpha\gamma} - \frac{1-J}{\int_J^1(a(j)-a(J))dj}\right]\frac{a_J}{a} \end{array} \right\}.$$

Consider a function

$$a\left(j\right) = \begin{cases} j, & \text{if } j \leq J_1 \\ J_1 + b\left(\frac{j-J_1}{J_2-J_1}\right)^N, & \text{if } J_1 < j \leq J_2 \\ \left(J_1+b\right)\left(\frac{j}{J_2}\right)^{1/N}, & \text{if } j > J_2. \end{cases} \tag{27}$$

where $b, N > 0$. Then,

$$\begin{aligned} \lim_{N\to\infty} a_J &= 0, \\ \lim_{N\to\infty} a &= J_1, \\ \lim_{N\to\infty} \bar{a} &= J_1 + \frac{1-J_2}{1-J}b, \\ \lim_{N\to\infty} \bar{a}_J &= \frac{1-J_2}{(1-J)^2}b. \end{aligned}$$

Therefore, as long as $\lambda > 1$, $\lim_{N\to\infty}\Phi_J > 0$. It follows that there exists a finite N for which $\Phi_J > 0$. ■

A.3 Proof of Proposition 3

Linearizing (13) about a steady state:

$$\begin{bmatrix} \hat{K}_{t+1} \\ \hat{C}_{t+1} \end{bmatrix} = \begin{bmatrix} Y'+1-\delta & -1 \\ \frac{\sigma R'}{R}C\left(Y'+1-\delta\right) & 1-\frac{\sigma R'}{R}C \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{C}_t \end{bmatrix}$$

The eigenvalues of the transition matrix are given by:

$$\xi_{1,2} = 1 + \frac{\left(Y'-\delta\right) - \frac{\sigma CR'}{R} \pm \sqrt{\left[\left(Y'-\delta\right)-\frac{\sigma CR'}{R}\right]^2 - 4\frac{\sigma CR'}{R}}}{2}.$$

If $R' < 0$ (odd steady states) both eigenvalues are real and $\xi_1 < 1 < \xi_2$. If $R' > 0$ (even steady states) there are four possible cases:

Condition	Steady state stability
$R' < 0$	saddle
$Y' - \delta > \frac{\sigma C R'}{R} > 0$ $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 > 4\frac{\sigma C R'}{R}$	source
$Y' - \delta > \frac{\sigma C R'}{R} > 0$ $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 < 4\frac{\sigma C R'}{R}$	unstable spiral
$\frac{\sigma C R'}{R} > Y' - \delta > 0$ $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 > 4\frac{\sigma C R'}{R}$	sink
$\frac{\sigma C R'}{R} > Y' - \delta > 0$ $\left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 < 4\frac{\sigma C R'}{R}$	stable spiral

Table A.1: Steady state stability for different parameters configurations

1. $Y' - \delta > \frac{\sigma C R'}{R} \wedge \left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 > 4\frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{R}, \left\|\xi_{1,2}\right\| > 1;$
2. $Y' - \delta > \frac{\sigma C R'}{R} \wedge \left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 < 4\frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{C}, \left\|\xi_{1,2}\right\| > 1;$
3. $Y' - \delta < \frac{\sigma C R'}{R} \wedge \left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 > 4\frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{R}, \left\|\xi_{1,2}\right\| < 1;$
4. $Y' - \delta < \frac{\sigma C R'}{R} \wedge \left[(Y' - \delta) - \frac{\sigma C R'}{R}\right]^2 < 4\frac{\sigma C R'}{R} \Rightarrow \xi_{1,2} \in \mathbb{C}, \left\|\xi_{1,2}\right\| < 1.$

■