Learning Dynamics: Tools and Conceptual Issues

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Motivation

- Two foundations of rational expectations equilibrium analysis
  - Optimization
  - Mutual consistency of beliefs
- Strong knowledge assumptions
  - Agents know more than the modeler

Scientific Method

- Principle components
  - Maintained theories
  - Procedures for collecting data
  - Procedures for confronting theory with data
  - Procedures for updating theory given discrepancy between theory and data

Bounded Rationality: Learning Dynamics

- Attempts to place the agent and modeler on equal footing

- Agents might proceed as:

  - Classical econometrician: know model but unsure about parameters

  - Bayesian econometrician: unsure about model and parameters but understand how they are unsure

Learning Dynamics

- Analyze broad class of model that relaxes the second pillar of REE analysis
  - Retain optimization
  - Permit non-rational expectations

- Beware of theorists bearing free parameters!
  - Constrain the analysis by requiring the procedure to nest rational expectations

Why learning?

- Permits analysis of a broader class of problems

- May resolve/elucidate
  - Questions of indeterminacy
  - Questions of robustness
  - Questions of dynamics
  - Questions of regime change

Lecture Objectives: Lecture 1

- Mathematical foundations
  - Learning renders model dynamics self-referential
  - Requires new set of tools to analyze model properties
- Conceptual issues in modeling multi-period decision problems

Lecture Objectives: Lecture 2

- Applications
  - Central Bank communication
  - Empirical modeling of macroeconomic dynamics
  - Policy design and interaction

A Simple Model

- Consider a variant of the Cobweb model

$$p_t = \mu + \alpha E_{t-1} p_t + \delta w_{t-1} + \eta_t \quad (1)$$

where $\mu,\ a,\ \delta > 0$; w_t and exogenous process; and η_t a bounded iid disturbance

- Unique bounded solution

$$p_t = \bar{a} + \bar{b} w_{t-1} + \eta_t$$

where

$$\bar{a} = (1-\alpha)^{-1} \mu \text{ and } \bar{b} = (1-\alpha)^{-1} \delta$$

Statistical Learning

- Suppose agents forecast p_t as a linear function

$$\hat{E}_{t-1} p_t = a_{t-1} + b_{t-1} w_{t-1} \tag{2}$$

  where $\{a_{t-1}, b_{t-1}\}$ are currently maintained beliefs

- Price dynamics then

$$p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1}) \, w_{t-1} + \eta_t$$

  - Self-referential system
  - Data generating process is non-stationary

Least Squares Learning

- Beliefs $\{a_{t-1}, b_{t-1}\}$ estimates using data $\{p_i, w_i\}_{i=0}^{i=t-1}$

- Ordinary least squares implies

$$\binom{a_{t-1}}{b_{t-1}} = \left(\sum_{i=1}^{t-1} z_{i-1} z'_{i-1}\right)^{-1} \left(\sum_{i=1}^{t-1} z_{i-1} p_i\right) \tag{3}$$

where $z_i = [1 \;\; w_i]$

  – Fully specified system: relations (1) - (3)

Question

- Under what conditions do $a_t \longrightarrow \bar{a}$ and $b_t \longrightarrow \bar{b}$ as $t \longrightarrow \infty$?

Theorem

- Consider the model (1) - (3). If $\alpha < 1$ then $(a_t, b_t) \rightarrow \left(\bar{a}, \bar{b}\right)$ with probability one. If $\alpha > 1$ then convergence occurs with probability zero.
  - Property $a < 1$ referred to as expectational stability
  - E-stability principle governed by mapping between agent beliefs and true model coefficients

E-stability principle

- Recall

$$\hat{E}_{t-1}p_t = a_{t-1} + b_{t-1}w_{t-1} \tag{4}$$

and

$$p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1})\, w_{t-1} + \eta_t$$

- Latter implies optimal rational forecast

$$E_{t-1}p_t = (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1})\, w_{t-1}$$

Mapping Defined

- Agents beliefs and optimal forecast define the mapping

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \mu + \alpha a \\ \delta + \alpha b \end{pmatrix}$$

  - An REE is a fixed point of this mapping
  - Application of method of undetermined coefficients

E-stability principle II

- Define the associated ordinary differential equation

$$\frac{d}{d\tau}\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} \mu + \alpha a \\ \delta + \alpha b \end{pmatrix}$$

  where τ is notional time

  - Local stability properties govern E-stability principle

E-stability principle III

- In particular, the REE is E-stable if and only if the ODE is local stable at $\left(\bar{a}, \bar{b}\right)$

- Hence

$$
\begin{aligned}
\frac{da}{d\tau} &= \mu + (\alpha - 1)\, a \\
\frac{db}{d\tau} &= \delta + (\alpha - 1)\, b
\end{aligned}
$$

requires $\alpha < 1$

Recursive Least Squares

- Ordinary least squares regression can be written as

$$
\begin{aligned}
\phi_t &= \phi_{t-1} + t^{-1} R_t z_{t-1} \left(p_t - \phi'_{t-1} z_{t-1} \right) \\
R_t &= R_{t-1} + t^{-1} \left(z_{t-1} z'_{t-1} - R_{t-1} \right)
\end{aligned}
$$

where $\phi_t = [a_t \; b_t]'$

Recursive Least Squares II

- Since

$$
\begin{aligned}
p_t &= (\mu + \alpha a_{t-1}) + (\delta + \alpha b_{t-1})\, w_{t-1} + \eta_t \\
&= T\left(\phi_{t-1}\right)' z_{t-1} + \eta_t
\end{aligned}
$$

we have

$$
\begin{aligned}
\phi_t &= \phi_{t-1} + t^{-1} R_t^{-1} z_{t-1} \left(z'_{t-1} \left(T\left(\phi_{t-1}\right) - \phi_{t-1} \right) + \eta_t \right) && (5) \\
R_t &= R_{t-1} + t^{-1} \left(z_{t-1} z'_{t-1} - R_{t-1} \right) && (6)
\end{aligned}
$$

– Question: does this system converge??

Stochastic Approximation Methods

- Provide results characterizing convergence of such systems
  - Ljung (1977), Marcet and Sargent (1989)

- Consider stochastic recursive algorithm

$$\theta_t = \theta_{t-1} + \gamma_t Q\left(t, \theta_{t-1}, X_t\right)$$

where

θ_t - parameter estimates (a_t, b_t, R_t)
X_t - state vector (effects of p_t, z_t and η_t)
γ_t - deterministic sequence of gains (t^{-1})

- Stochastic approximation approach associates the ODE

$$\frac{d\theta}{d\tau} = h\left(\theta\left(\tau\right)\right)$$

where

$$h\left(\theta\right) \equiv \lim_{t \to \infty} EQ\left(t, \theta, X_t\right)$$

  - E denotes the expectation of $Q\left(t, \theta, X_t\right)$ with respect to the invariant distribution of X_t for fixed θ

Stochastic Approximation Results

- Under suitable assumptions:
  - If $\bar{\theta}$ is a locally stable equilibrium pint of the ODE, then $\bar{\theta}$ is a possible point of convergence of the SRA
  - If $\bar{\theta}$ is not a locally stable equilibrium point of the ODE, then $\bar{\theta}$ is not a possible point of convergence of the SRA. That is $\theta_t \rightarrow \bar{\theta}$ with probability zero.

Technical Assumptions

- Stochastic approximation results rely on assumptions regarding
  - Regularity conditions on Q
  - Conditions on the rate at which $\gamma_t \longrightarrow 0$
  - Assumptions on the properties of X_t
    - Evans and Honkapohja (2001, Chapter 6 & 7)

Example Revisited

- To write our RLS algorithm in SRA form define $S_{t-1} \equiv R_t$ and write (5) and (6) as

$$\begin{aligned}
\phi_t &= \phi_{t-1} + t^{-1} S_{t-1}^{-1} z_{t-1} \left(z'_{t-1} \left(T \left(\phi_{t-1} \right) - \phi_{t-1} \right) + \eta_t \right) \\
S_t &= S_{t-1} + t^{-1} \left(\frac{t}{t+1} \right) \left(z_t z'_t - S_{t-1} \right)
\end{aligned}$$

so that

$$\begin{aligned}
\theta_t &= \text{vec}\left(\phi_t, S_t \right) \\
X_t &= [1 \; w_t \; w_{t-1} \; \eta_t]' \\
\gamma_t &= t^{-1}
\end{aligned}$$

Example Revisited II

- Hence

$$\begin{aligned} Q_\phi(t, \theta_{t-1}, X_t) &= S_{t-1}^{-1} z_{t-1} \left(z'_{t-1} \left(T\left(\phi_{t-1}\right) - \phi_{t-1}\right) + \eta_t \right) \\ Q_S(t, \theta_{t-1}, X_t) &= vec\left(\left(\frac{t}{t+1} \right) \left(z_t z'_t - S_{t-1} \right) \right) \end{aligned}$$

  - To do: compute the associated ODE!
  - Fix θ_t and compute expectation over X_t

Example Revisited III

- Fixing (ϕ, S) implies

$$
\begin{aligned}
h_{\phi}(\phi, S) &= \lim_{t \to \infty} E S^{-1} z_{t-1} \left(z'_{t-1} \left(T(\phi) - \phi \right) + \eta_t \right) \\
h_S(\phi, S) &= \lim_{t \to \infty} E \left(\frac{t}{t+1} \right) \left(z_t z'_t - S \right)
\end{aligned}
$$

- Since

$$
E z_t z'_t = E z_{t-1} z'_{t-1} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \equiv M
$$

where $\Omega = E\left[w_t w'_t\right]$; $E z_{t-1} \eta_t = 0$ and $\lim_{t \to \infty} \frac{t}{t+1} = 1$

$$
\begin{aligned}
h_{\phi}(\phi, S) &= S^{-1} M \left(T(\phi) - \phi \right) \\
h_S(\phi, S) &= M - S
\end{aligned}
$$

- The Associated ODE is then

$$\frac{d\phi}{d\tau} = S^{-1}M\left(T\left(\phi\right)-\phi\right)$$
$$\frac{dS}{d\tau} = M - S$$

  - The latter is globally stable for any initial S: $S \rightarrow M$
  - Therefore $S^{-1}M \rightarrow I$

- Hence

$$\frac{d\phi}{d\tau} = T\left(\phi\right)-\phi$$

  - Intuition?

Summary

- Have motivated the concept of E-Stability
  - Associated ODE
- Remainder
  - Microfoundations and modeling issues
  - Application in policy design

A Familiar Example

- New Keynesian model

$$x_t = E_t x_{t+1} - \sigma \left(i_t - E_t \pi_{t+1} - r_t \right)$$

$$\pi_t = k x_t + \beta E_t \pi_{t+1}$$

where $0 < \beta < 1$ and $k, \sigma > 0$.

A Familiar Example II

- Is the correct analogue under learning

$$x_t = \hat{E}_t x_{t+1} - \sigma \left(i_t - \hat{E}_t \pi_{t+1} - r_t \right)$$

$$\pi_t = k x_t + \beta \hat{E}_t \pi_{t+1}$$

where

$$\hat{E}_t z_{t+1} = a_{t-1} + b_{t-1} z_t$$

and $z_t = [x_t \ \pi_t \ r_t]'$

Modeling Learning Dynamics: Some Issues

- Intertemporal optimization implies multi-period forecasts
  - Fundamental to macroeconomic analysis

- Intertemporal budget constraints central to this theory
  - Partial Equilibrium: Marcet and Sargent (1989)
  - General Equilibrium: Preston (2005)

Model Agents

- Households
- Firms
- Monetary authority
- Fiscal authority

Model Features

- Money or cashless limit
- Monopolistic competition/nominal rigidities
- Incomplete asset markets
- No capital
- Non-rational expectations

Beliefs

Under rational expectations:

1. Agents optimize given beliefs

2. The probabilities they assign to future state variables coincide with the predictions of the model

This paper retains (1) and replaces (2) with

$2'$. Future state variables outside agent's control are forecasted using an econometric model.

Knowledge

- Know own preferences and constraints

- Do not know true economic model of determination of variables outside their control.

  - E.g. even if $Y_t^i = Y_t^j = Y_t$ in EQ $\not\Rightarrow$ agents know $Y_t^i = Y_t^j$

- Observe aggregate variables and disturbances

- Forecast variables outside their control using an atheorectical VAR

  - Takes the minimum state variable form

Household Problem

- Households seek to maximize

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[U\left(C_T^i; \xi_T\right) - v\left(h_T^i; \xi_T\right)\right]$$

subject to

$$\begin{aligned} B_t^i &\leq (1 + i_{t-1}) B_{t-1}^i + w_t h_t^i + \int_0^1 \Pi_t(j)\, dj - T_t - P_t C_t^i \\ &= (1 + i_{t-1}) B_{t-1}^i + PY_t^i - T_t - P_t C_t^i \end{aligned}$$

  - Beliefs: a-theoretical VAR of exogenous variables
  - Nests MSV REE

First order conditions

- Log linear approximation yields

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma \left(i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t g_{t+1} \right)$$

and

$$\hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{C}_T^i = A_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \hat{Y}_T^i$$

where

$$A_t^i = B_t^i / P_t \bar{Y}; \qquad g_t = \xi_t u_{c\xi} / u_c; \qquad \sigma^{-1} = -u_{cc} \bar{Y} / u_c$$

Optimal decision rule

- First order conditions imply

$$\hat{C}_t^i = (1-\beta) A_t^i + \hat{E}_t^i \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta) \hat{Y}_T^i - \beta\sigma (\hat{\imath}_T - \pi_{T+1}) + \beta (g_T - g_{T+1}) \right]$$

  - Forecasts about prices into the indefinite future matter!

Comparison to Existing Approaches

- Assume $A_t^i = 0$ and quasi difference optimal decision rule

$$\hat{C}_t^i = (1-\beta)\,\hat{Y}_t - \beta\sigma\left(\hat{\imath}_t - \hat{E}_t\pi_{t+1}\right) + \beta\left(g_t - \hat{E}_t g_{t+1}\right) + \beta\hat{E}_t\hat{C}_{t+1}^i$$

- Assuming market clearing conditions known: i.e: $C_t^i = C_t = Y_t$ gives

$$\hat{C}_t^i = \hat{E}_t^i\hat{C}_{t+1}^i - \sigma\left(i_t - \hat{E}_t^i\pi_{t+1} - g_t + \hat{E}_t g_{t+1}\right)$$

Comparison to Existing Approaches II

- Are decisions implied by the optimal decision equivalent to those implied by

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma \left(i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t g_{t+1} \right)$$

  – Where forecasts determined by the model

$$z_t^* = a_{t-1} + b_{t-1} z_{t-1} + \varepsilon_t$$

with $z_t^* = \left[\hat{C}_t^i \; \pi_t \; i_t \; g_t \right]'$ and $z_t = \left[\hat{Y}_t \; \pi_t \; i_t \; g_t \right]'$

Comparison to Existing Approaches II

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$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \sigma \left(i_t - \hat{E}_t^i \pi_{t+1} - g_t + \hat{E}_t g_{t+1} \right)$$

  – Where forecasts determined by the model

$$z_t^* = a_{t-1} + b_{t-1} z_{t-1} + \varepsilon_t$$

  with $z_t^* = \left[\hat{C}_t^i \; \pi_t \; i_t \; g_t \right]'$ and $z_t = \left[\hat{Y}_t \; \pi_t \; i_t \; g_t \right]'$

- In general - No!

- Optimal decision rule if expectations treated correctly

  – i.e. the conditional expectation of consumption induced by model primitives and optimality

Forecasting Consumption

- Assuming $A_t^i = 0$ for all i the optimal forecast

$$\hat{E}_t^i \hat{C}_{t+1}^i = \hat{E}_t^i \sum_{T=t+1}^{\infty} \beta^{T-t} \left[(1-\beta) \hat{Y}_T^i - \beta\sigma \left(\hat{\imath}_T - \pi_{T+1} \right) + \beta \left(v_T - v_{T+1} \right) \right]$$

- Not the same as

$$\hat{E}_t^i z_{t+1}^* = a_{t-1} + b_{t-1} z_t$$

Commentary

- Market clearing conditions are part of the REE

- Does not deliver optimal decision rule
  - Confuses endogenous decision variables and exogenous state variables
  - The correct distribution with respect to which expectations are taken is induced from the decision problem and properties of exogenous variables
  - In general intertemporal budget constraint will be violated in this formulation

Aggregate Consumption Dynamics

- Aggregating over the continuum provides

$$\hat{C}_t = \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\,\hat{Y}_T - \beta\sigma\left(\hat{\imath}_T - \pi_{T+1}\right) + \beta\left(g_T - g_{T+1}\right) \right]$$

  where

$$\hat{C}_t \equiv \int \hat{C}_t^i di; \quad \hat{E}_t \equiv \int \hat{E}_t^i di; \quad 0 \equiv \int A_t^i di$$

  - For any variable X_t, $\hat{E}_t \hat{E}_{t+1} X_{t+1} \neq \hat{E}_t X_{t+1}$

Summary

- Under rational expectations equilibrium probability laws satisfy all relevant constraints

- Under learning dynamics expectations about future endogenous decision variables are taken with respected to a distribution

  - Induced by beliefs about exogenous states

  - Induced by optimal decisions

Firms

- Continuum of firms maximize

$$\hat{E}_t^j \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left[\Pi_T^j \left(p_t(j) \right) \right]$$

where

$$\Pi_T^j (p) = Y_T P_T^{\theta} p^{1-\theta} - w_T^i f^{-1} \left(Y_T P_T^{\theta} p^{-\theta} / A_T \right)$$

$$y_t^i = A_t f \left(h_t^i \right)$$

Optimal Price Setting

- Log-linear approximation implies

$$p_t^j = \hat{E}_t^j \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{(1-\alpha\beta)\left(\omega + \sigma^{-1}\right)}{1+\omega\theta} x_T + \pi_T \right]$$

  - Infinite horizon forecasts matter
  - $x_t = Y_t - \hat{Y}_t^n$ is the output gap

Model Summary

- Aggregate demand and supply:

$$
\begin{aligned}
x_t &= \hat{E}_t \sum_{T=t}^{\infty} \beta^{T-t} \left[(1-\beta)\, x_{T+1} - \sigma \left(i_T - \pi_{T+1} - r_T \right) \right] \\
\pi_t &= \kappa x_t + \hat{E}_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\kappa \alpha \beta x_{T+1} + (1-\alpha)\beta \pi_{T+1} \right]
\end{aligned}
$$

Rational Expectations

- Under this assumption $\hat{E}_t \hat{E}_{t+1} X_{t+1} = \hat{E}_t X_{t+1}$. Hence

$$
\begin{aligned}
x_t &= E_t\left[(1-\beta)\, x_{t+1} - \sigma\left(i_t - \pi_{t+1} - r_t\right)\right] + \\
&\quad \hat{E}_t \sum_{T=t+1}^{\infty} \beta^{T-t}\left[(1-\beta)\, x_{T+1} - \sigma\left(i_T - \pi_{T+1} - r_T\right)\right] \\
&= E_t\left[(1-\beta)\, x_{T+1} - \sigma\left(i_t - \pi_{t+1} - r_t\right)\right] + \beta E_t x_{t+1} \\
&= E_t x_{t+1} - \sigma\left(i_t - E_t \pi_{t+1} - r_t\right)
\end{aligned}
$$

- Rational expectations analysis gives

$$\begin{aligned} x_t &= E_t x_{t+1} - \sigma\left(i_t - E_t \pi_{t+1} - r_t\right) \\ \pi_t &= k x_t + \beta E_t \pi_{t+1} \end{aligned}$$

- Bullard and Mitra (2002) and Evans and Honkapohja (2003)

$$\begin{aligned} x_t &= \hat{E}_t x_{t+1} - \sigma\left(i_t - \hat{E}_t \pi_{t+1} - r_t\right) \\ \pi_t &= k x_t + \beta \hat{E}_t \pi_{t+1} \end{aligned}$$

Does any of this matter?

Application: Policy Design

- Early literature concerned with robustness and equilibrium selection

- Models with learning also permit addressing critiques of policy
  - Friedman (1968)

An Interest Rate Peg

- Consider the policy $i_t = \phi_r r_t$
  - Indeterminacy: Sargent and Wallace (1975)

- Agent forecasts given by

$$\hat{E}_t z_T = a_{t-1} + b_{t-1} \rho^{T-t} r_t$$

for $z_t = [\pi_t \; x_t \; i_t]'$. Combined with structural model gives mapping

$$\begin{aligned} \frac{da}{d\tau} &= T(a) - a \\ \frac{db}{d\tau} &= T(b) - b \end{aligned}$$

Stability AnalysisStability Analysis

- Straightforward to show under optimal decisions

$$T'(a) = \begin{bmatrix} \frac{(1-\alpha)\beta}{1-\alpha\beta} + \frac{\kappa\sigma}{1-\beta} & \frac{\kappa\alpha\beta}{1-\alpha\beta} + \kappa & -\frac{\kappa\alpha\beta}{1-\alpha\beta} \\ \frac{\sigma}{1-\beta} & 1 & -\frac{\sigma\beta}{1-\beta} \\ 0 & 0 & 0 \end{bmatrix}$$

  and

$$\det(T'(a) - I) = \frac{\kappa\sigma}{1-\beta} + \frac{\alpha\beta\kappa\sigma}{(1-\beta)(1-\alpha\beta)} > 0$$

  - Expectations driven fluctuations
  - True for the Euler equation approach: Evans and Honkapohja (2003)

Taylor Rule

- McCallum (1983): conditioning on endogenous variables resolves indeterminacy

- Consider the rule

$$i_t = \phi_\pi \pi_t + \phi_x x_t$$

  - Requires

$$\kappa \left(\phi_\pi - 1\right) + \left(1 - \beta\right) \phi_x > 1$$

    for determinacy under REE and stability under learning

  - Bullard and Mitra (2002), Preston (2005)

Emerging differences

- Policy actions that depend on misspecified beliefs can lead to self-fulfilling expectations

- For instance

$$i_t = \phi_\pi \hat{E}_t \pi_{t+1}$$

  - Optimal decisions requires

$$\phi_\pi > \frac{1}{1-\beta} - \frac{\alpha\left(2-\beta-\alpha\beta\right)}{\left(1-\alpha\right)\left(1-\alpha\beta\right)^2}$$

  - Euler equation approach requires $\phi_\pi > 1$

Raises important questions

- What is the appropriate use of forecasts in policy design?

- Can instability be mitigated?

  – The role of communication

  – Optimal policy/targeting rules imply a particular kind of dependency of interest rate decisions on forecasts

- Different statistical properties

  – Forecasts, dynamics and policy evaluation