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Econ 503 Homework 3
Answers

1. (10.C.3)

   (a) The first order conditions are $c_j'(q_j^*) \geq \lambda$ for all j, and hold with equality if $q_j^* > 0$, where λ is the Lagrange multiplier.

   (b) By the Envelope theorem $C'(q) = \lambda$ since q appears only in the constraint.

   (c) Firm j solves

   $$\max_{q_j \geq 0} pq_j - c_j(q_j).$$

   Let $p = C'(q)$. The first order conditions are $C'(q) = c_j'(q_j)$ for $q_j > 0$. If we assume strict convexity and strict monotonicity then $q_j = q_j^*$ and hence the supply sunction is $q(p) = \sum q_j = \sum q_j^* = q$. That implies $C'(q)$ is the inverse industry supply function.

2. (a) Assume that $F^1(K^1, L^1)$, $F^2(K^2, L^2)$ are continuous, strictly concave, and strictly increasing. The production possibility frontier is solved by

   $$Y^1 = g(K, L, I) = \max F^1(K^1, L^1)$$

   s.t.

   $$0 \leq K^1 \leq K$$
   $$0 \leq L^1 \leq L$$
   $$F^2(K - K^1, L - L^1) \geq I.$$

To show concavity, let $(K^\lambda, L^\lambda, I^\lambda) = \lambda(K, L, I) + (1-\lambda)(K', L', I')$ for $0 < \lambda < 1$. Let (K^1, L^1) and $(K^{1'}, L^{1'})$ solves $g(K, L, I)$ and $g(K', L', I')$ respectively. Finally let $(K^{1\lambda}, L^{1\lambda}) = \lambda(K^1, L^1) + (1-\lambda)(K^{1'}, L^{1'})$. Then we have

$$0 \le K^{1\lambda} \le K^\lambda,$$
$$0 \le L^{1\lambda} \le L^\lambda.$$

Moreover, by concavity of $F^1(K^1, L^1)$ and $F^2(K^2, L^2)$,

$$\begin{aligned} F^2(K^\lambda - K^{1\lambda}, L^\lambda - L^{1\lambda}) &\ge I^\lambda, \\ g(K^\lambda, L^\lambda, I^\lambda) \ge F^1(K^{1\lambda}, L^{1\lambda}) &\ge \lambda g(K, L, I) + (1-\lambda) g(K', L', I'). \end{aligned}$$

(b) To show that $g(K, L, I)$ is increasing with K and L, let $(K', L') \ge (K, L)$ and (K^{1*}, L^{1*}) solves $g(K, L, I)$. Then

$$g(K', L', I) \ge F^1(K' - K + K^{1*}, L' - L + L^{1*}) \ge F^1(K^{1*}, L^{1*}) = g(K, L, I).$$

To show that $g(K, L, I)$ is decreasing with I, let $I' < I$. Then there exist $\epsilon > 0$ such that

$$F^2(K - K^{1*} - \epsilon, L - L^{1*} - \epsilon) > I'.$$

Then $g(K, L, I') \ge F^1(K^{1*} + \epsilon, L^{1*} + \epsilon) > g(K, L, I)$.

(c) Assume that $F^1(K^1, L^1)$ and $F^2(K^2, L^2)$ are linearly homogeneous and differentiable. Then from the envelope theorem and the first order conditions of the above problem we get

$$\begin{aligned}
\frac{\partial g(K,L,I)}{\partial K} &= \lambda F_1^2(K-K^1, L-L^1) = F_1^1(K^1, L^1),\\
\frac{\partial g(K,L,I)}{\partial L} &= \lambda F_2^2(K-K^1, L-L^1) = F_2^1(K^1, L^1),\\
\frac{\partial g(K,L,I)}{\partial I} &= -\lambda.
\end{aligned}$$

On the other hand, in a competitive market the consumption goods sector's objective is

$$\max_{K^1,L^1} F^1(K^1, L^1) + q(1-\mu)K^1 - wL^1 - rK^1,$$

and the investment goods sector's objective is

$$\max_{K^2,L^2} q[F^2(K^2, L^2) + (1-\mu)K^2] - wL^2 - rK^2,$$

where r, w, q are the price of capital (rental rate), labor (wage rate), and investment respectively. So the first order conditions for firms are

$$\begin{aligned}
r &= F_1^1(K^1, L^1) + q(1-\mu) = q(F_1^2(K^2, L^2) + 1 - \mu),\\
w &= F_2^1(K^1, L^1) = qF_2^2(K^2, L^2),
\end{aligned}$$

Thus given $g(K, L, I)$, we can solve

$$\begin{aligned}
q &= \lambda = -\frac{\partial g(K,L,I)}{\partial I},\\
r &= \frac{\partial g(K,L,I)}{\partial K} + q(1-\mu),\\
w &= \frac{\partial g(K,L,I)}{\partial L}.
\end{aligned}$$

(d) Let K^{1*}, L^{1*} solves $g(K, L, I) = F^1(K^{1*}, L^{1*})$. The ratio of factors used in the two sector are $\frac{K^{1*}}{L^{1*}}$, $\frac{K^{2*}}{L^{2*}} = \frac{K-K^{1*}}{L-L^{1*}}$, respectively. However, without further assumptions, we cannot decide which ratio is bigger.

(e) **Factor intensity assumption:** Suppose that one good (good 1) is relatively more intensive in labor than the other good (good 2), i.e.

$$\frac{a_{L1}(p)}{a_{K1}(p)} > \frac{a_{L2}(p)}{a_{K2}(p)}, \text{ for all } p = (w, r),$$

where $a_{ij}(p)$ is the good i needed to produce 1 unit of good j, given price vector p and cost minimization.

Stolper-Samuelson Theorem: Given the factor intensity assumption, if the price of output j increases, then given interior equilibria both before and after the price change, the equilibrium price of the factor more intensively used in the production of good j increases, while the price of the other factor decreases.[1]

3. (a) In the competitive equilibrium, the firm's objective is

$$\max_{L_t, K_t} F(K_t, L_t) - w_t L_t - r_t K_t$$

Assume that $F(K, L)$ is increasing, concave, and linearly homogeneous. Note that at the optimal choice of (K_t, L_t), $r_t = F_1(K_t, L_t)$, $w_t = F_2(K_t, L_t)$, and $F(K_t, L_t) = w_t L_t + r_t K_t$.

The consumer's objective is

$$\max_{c_t, K_{t+1}} \sum_t \delta^t u(c_t)$$

s.t.

$$c_t + K_{t+1} \leq w_t L_t + r_t K_t, \;\; L_t = 1 \; \forall t, k_0 \text{ given.}$$

[1] Check MWG 15.D for discussions.

Note that using firm's optimality conditions, $\{c_t, K_{t+1}\}$ also solves the planner's problem:

$$W(K_0) = \max_{c_t, K_{t+1}} \sum_t \delta^t u(c_t)$$

s.t.

$$c_t + K_{t+1} \leq F(K_t, 1)\ \forall t, k_0 \text{ given.}$$

(b) Note that by principle of maximality, the planner's problem is equivalent to solving the Bellman equation

$$W(K) = \max_{c,K'} u(c) + \delta W(K'),\ \text{s.t. } c + K' \leq F(K, 1).$$

Note that the right-hand side of the equation is a standard consumer choice problem between choosing consumption and capital investment.

(c) If $K' > K$, then in the next period the consumer solves

$$W(K') = \max_{c',K''} u(c') + \delta W(K''),\ \text{s.t. } c' + K'' \leq F(K', 1).$$

Note that monotonicity of u implies $c + K' = F(K, 1) < F(K', 1) = c' + K''$, and strict concavity of u implies W is strictly concave. Then from the first order conditions $u'(c) = \delta W'(K')$ and $u'(c') = \delta W'(K'')$ we can show $K'' > K'$. The intuition of this result is that both consumption and investment are normal goods, and so when the budget constraint relaxes the levels of both goods increase. Similarly if $K' < K$

then $K'' < K'$.

(d) For decreasing K_t, since K_t is bounded below by 0, it will converge to some K^*. On the other hand, for increasing K_t it can either converge to a finite level or has unbounded growth. Note that from the Euler equation $u'(c_t) = \delta u'(c_{t+1}) F_1(K_{t+1}, 1)$, for increasing K_t, we will also have increasing c_t and hence $1 < \delta F_1(K_{t+1}, 1)$, given that $u'(c) > 0$. Since the marginal productivity of capital $F_1(K, 1)$ is decreasing with K, the necessary and sufficient condition for converging K_t is $\lim_{K \to \infty} F_1(K, 1) < 1/\delta$, given that $F_1(0, 1) > 1/\delta$.