Econ 4111
Professor: John Nachbar
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# Set Theory

## 1 Overview

This is an informal introduction to Set Theory, which is somewhat ironic because set theory is, by its nature, a highly formal subject. I try to convey the main line of development without devoting too much space to technicalities. I follow largely Halmos (1970). More formal references are Suppes (1960) and Enderton (1977).

I have two objectives. The first is to introduce basic concepts like cardinality, which I have to cover anyway, in a format that is relatively entertaining. The second is to demonstrate both the necessity and the possibility of building up mathematics in a highly systematic way.

Set Theory does not include a formal definition of what it means for $x$ to be an element of a set $A$, written $x \in A$. Implicitly, Set Theory assumes that this concept already has meaning for us. Set Theory focuses instead on rules for forming sets and on the way that sets can be used to construct other objects of interest to us, such as functions and numbers.

## 2 The Axioms of Set Theory

The theorems collectively known as Set Theory can be axiomatized in a number of different ways, but the standard axiomatization is Zermelo-Fraenkel (ZF) Set Theory. Without bogging down in the details of the subject, I take the axioms to be the following.

1. **Existence.**
2. **Extension.**
3. **Specification.**
4. **Pairing.**
5. **Union.**
6. **Power.**
7. **Infinity.**
8. **Regularity.**

9. **Choice.**

10. **Replacement.**

I do not discuss Replacement, which is highly technical. And I mention Regularity only in passing, in Section 1. All of the others, however, I discuss in some detail.

# 3 Axiom: Existence

*There exists a set.*

This axiom seems silly but one of the points of Set Theory is to be very careful as to what one allows to be called a set. The reason for this will become clear after I introduce Theorem 1. Existence is implied collectively by some of the other axioms, so that an explicit axiom is not needed. But having an explicit axiom makes for a cleaner exposition.

# 4 Axiom: Extension

## 4.1 Statement of the Axiom of Extension.

*If $A$ and $B$ are sets then $A = B$ iff, for every $x$, $x \in A$ iff $x \in B$.*

A subtle point is that I am implicitly assuming a background logic in which equality already has a basic meaning. So, even without Extension, I can write $A = A$. A formal treatment of Set Theory would be more explicit about this sort of thing.

What Extension brings that is new is that two sets are equal if they have the same elements, even if the sets appear to differ in some other way, such as in the way in which the sets are described. Extension says that, so far as Set Theory is concerned, the *only* attribute of a set that matters is its elements.

## 4.2 Some standard set theory notation.

$\{a, b\}$ is understood to mean the set with elements $a$ and $b$. By Extension, $\{a, a\} = \{a\}$ and $\{a, b\} = \{b, a\}$.

Write $A \subseteq B$ iff $x \in A$ implies $x \in B$. Thus Extension implies that $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

If $A \subseteq B$ but there is an $x \in B$ such that $x \notin A$ then write $A \subset B$. Here and elsewhere, a strikethrough ($\notin$, $\neq$, $\not\subseteq$, $\not\subset$) indicates negation.

# 5 Axiom: Specification

## 5.1 Statement of the Axiom of Specification.

> *If $A$ is a set and, for every $x \in A$, $P(x)$ is a statement about $x$, then there is a set $B$ such that $x \in B$ iff $x \in A$ and $P(x)$ is true.*

I write,

$$B = \{x \in A : P(x)\}.$$

For example, I could use Specification to identify the set of cats that have white paws. $A$ would be the set of all cats. $P(x)$ would be read: "cat $x$ has white paws."

There are rules governing what can constitute a valid $P$. I won't go into what these rules are, because the material quickly becomes quite technical, but the effect is to exclude $P$ that are gibberish, vague, or paradoxical. In informal examples, such as the one above, there may be some ambiguity about $P$ (how white do paws have to be to be considered white?). In the formal development, $P$ will be unambiguous.

## 5.2 The Empty Set.

Let $A$ be any set (one such set exists, by Existence) and let

$$B = \{x \in A : x \neq x\}.$$

Therefore, $B$ has no elements. By Extension, there is only one set with no elements (trivially, since any two such sets have the "same" elements). This unique empty set is denoted $\emptyset$.

Trivially, for any set $A$, $\emptyset \subseteq A$.

## 5.3 There is no "Set of All Sets."

Specification alone, without any other assumptions, gives the following remarkable result.

**Theorem 1.** *If $A$ is a set then there is a set $B$ such that $B \notin A$.*

I provide a proof below. In words, Theorem 1 says that there is no set of all sets, no universal set. In the early days of Set Theory, it was naively assumed that *was* a universal set: one can sort of imagine all sets, and then one imagines taking the set of all of that. Existence of a set of all sets led, however, to a conundrum called Russell's Paradox that centers on the question: is the universal set an element of itself?

Set Theory avoids paradox by *not* assuming the existence of a universal set and then showing, via Theorem 1, that such a set cannot, in fact, exist. Russell's Paradox supplies the main idea in the proof. Theorem 1 implies that if I were to axiomatically assume existence of such a set then Set Theory would be inconsistent, and therefore useless: every statement would be both true and false. Theorem 1 also provides a vivid demonstration that we can *not* take existence of sets for granted. Any existence statement must be justified either directly or indirection (via proof) from the Set Theory axioms.

Since there is no universal set, it is a good idea to be careful and write $\{x \in A : P(x)\}$ rather than just $\{x : P(x)\}$. In many cases, the set $A$ is understood (perhaps because you have been using $A$ all along). But there is a danger of thinking that $A$ exists when it does not.

To avoid using the word "set" where it is not justified, I sometimes use the word "collection." Any grouping of objects can be a collection, and any set is a collection, but not all collections are sets. For example, I could refer to the collection of all sets, even though this collection is not a set. The distinction is not just words: we will be deriving theorems about sets and these theorems need not extend to all collections.

Before going over the proof, note that if there were a universal set then it would be a member of itself. Can any set be an element of itself? This seems weird and for a number of reasons Set Theory rules it out: the Axiom of Regularity implies that if $A$ is a set then $A \notin A$. Theorem 1, thus follows immediately from Regularity. But the proof does not *require* Regularity. The proof only requires Specification.

**Proof.** Define

$$B = \{x \in A : x \notin x\}.$$

Regularity implies that $B = A$ but if I do not assume Regularity then it is conceivable that $B$ is a proper subset of $A$.

I complete the proof with two different arguments that $B \notin A$, either of which suffices.

1. Consider any $x \in A$. If $x \in x$, then, by the definition of $B$, $x \notin B$. Then $x \neq B$ (since there is an element, namely $x$, that is in $x$ but not in $B$). On the other hand, if $x \notin x$, then, by the definition of $B$, $x \in B$. Again, $x \neq B$ (since there is an element, namely $x$, that is in $B$ but not in $x$). Thus, either way, $x \neq B$. Since $x$ was an arbitrary element of $A$, no element of $A$ is equal to $B$: $B \notin A$.

2. By *Reductio Ad Absurdum* (RAA). Suppose $B \in A$. If $B \in B$ then, by the definition of $B$, $B \notin B$. Conversely, if $B \notin B$ then, by the definition of $B$, $B \in B$. Thus, if $B \in A$ then $B \in B$ iff $B \notin B$. By RAA, $B \notin A$.

■

Either of the proofs just given is valid, but the RAA proof is the standard one.

# 6 Axiom: Pairing

## 6.1 Statement of the Axiom of Pairing.

*If $A$ and $B$ are sets then there is a set that contains $A$ and $B$ as elements.*

By Specification, Pairing implies that there exists a set, namely $\{A, B\}$, containing exactly $A$ and $B$ as elements

## 6.2 Some implications for the empty set.

An important consequence of Pairing is that $\{\emptyset\}$ is a set, since Pairing gives $\{\emptyset, \emptyset\}$ and Extension reduces this to just $\{\emptyset\}$.

Note that $\emptyset \neq \{\emptyset\}$: $\emptyset \in \{\emptyset\}$ whereas $\emptyset \notin \emptyset$ (since $\emptyset$ has no elements). In particular, Pairing implies the existence of a non-empty set, which we have constructed out of "nothing" (or, more accurately, out of the empty set).

# 7 Axiom: Union

## 7.1 Statement of the Axiom of Union.

*If $\mathcal{A}$ is a set of sets then there is a set $B$ such that $x \in B$ if there exists an $A \in \mathcal{A}$ such that $x \in A$.*

By Specification, Union implies that there exists a set, written

$$\bigcup_{A \in \mathcal{A}} A,$$

such that $x \in \bigcup_{A \in \mathcal{A}} A$ iff $x \in A$ for some $A \in \mathcal{A}$.

If $\mathcal{A}$ contains just two sets, say $\mathcal{A} = \{A_1, A_2\}$, then I write $A_1 \cup A_2$.

For example, suppose $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$. Then $A_1 \cup A_2 = \{1, 2, 3\}$. On the other hand, pairing gives $\{A_1, A_2\} = \{\{1, 2\}, \{2, 3\}\} \neq A_1 \cup A_2$

## 7.2 Intersection

Let $\mathcal{A}$ be a set of sets. Let $\hat{A} = \bigcup_{A \in \mathcal{A}} A$. By Union, $\hat{A}$ is a set. Define

$$B = \{x \in \hat{A} : \forall A \in \mathcal{A}, x \in A\}.$$

$B$ is the intersection of the sets in $\mathcal{A}$, written

$$B = \bigcap_{A \in \mathcal{A}} A.$$

### 7.3 Complements

Given a set $X$ and a set $A \subseteq X$, let the *complement* of $A$, written $A^c$, be defined by

$$A^c = \{x \in X : x \notin A\}.$$

A subtlety here is that $A^c$ must be defined with respect to a particular $X$. In practice, $X$ is usually fixed at the outset and $A$ is understood to be a subset of this $X$.

Given a set $\mathcal{A}$ of subsets of some set $X$, it is of interest to take the union or intersection of the corresponding complement sets. The standard notation is $\bigcup_{A\in\mathcal{A}} A^c$ and $\bigcap_{A\in\mathcal{A}} A^c$. We need to check that these sets are well defined. Formally, define

$$\bigcup_{A\in\mathcal{A}} A^c = \{x \in X : \exists A \in \mathcal{A} \text{ such that } x \in A^c\}$$

and

$$\bigcap_{A\in\mathcal{A}} A^c = \{x \in X : \forall A \in \mathcal{A}, x \in A^c\}.$$

**Theorem 2** (DeMorgan's laws)**.** *Let $X$ be a set and let $\mathcal{A}$ be a set of subsets of $X$. Then the following hold.*

1. $\left(\bigcup_{A\in\mathcal{A}} A\right)^c = \bigcap_{A\in\mathcal{A}} A^c$.
2. $\left(\bigcap_{A\in\mathcal{A}} A\right)^c = \bigcup_{A\in\mathcal{A}} A^c$.

**Proof.** For any $x \in X$, $x \in \left(\bigcup_{A\in\mathcal{A}} A\right)^c$ iff $x \notin \bigcup_{A\in\mathcal{A}} A$ iff $x \notin A$ for any $A \in \mathcal{A}$ iff $x \in A^c$ for every $A \in \mathcal{A}$ iff $x \in \bigcap_{A\in\mathcal{A}} A^c$. This shows that $\left(\bigcup_{A\in\mathcal{A}} A\right)^c = \bigcap_{A\in\mathcal{A}} A^c$. The other part of the proof is similar. ■

The above is an example of a straight deductive proof. Note that neither the theorem nor the proof require the sets to be nonempty.

## 8 Axiom: Power.

### 8.1 Statement of the Axom of Power.

*If $A$ is a set then there is a set $B$ containing all of the subsets of $A$.*

By Specification, Power implies that, in particular, there is a set containing precisely the subsets of $A$. This set is denoted $\mathbb{P}(A)$ and is called the *power set* of $A$.

## 8.2 Cartesian Products I.

Let $A$ and $B$ be non-empty sets. I want to construct the ordered pair $(a, b)$, with $a \in A$ and $b \in B$, as a set theoretic object. I can't use $\{a, b\}$ because $\{a, b\} = \{b, a\}$, by Extension. A construction that will work is $(a, b) = \{\{a\}, \{a, b\}\}$. I leave it to as an exercise to verify that $(a, b) = (\hat{a}, \hat{b})$ iff $a = \hat{a}$ and $b = \hat{b}$. The Cartesian Product

$$A \times B = \{x \in \mathbb{P}(\mathbb{P}(A \cup B)) : \exists a \in A, b \in B \text{ such that } x = \{\{a\}, \{a, b\}\}\}$$

Having constructed $(a, b)$ and $A \times B$ out of sets, one can now forget the details of the construction.

In Section 10.2, I discuss how to construct arbitrary Cartesian products (rather than just pairs).

## 8.3 Relations

A relation $R$ on $A \times B$ is simply *any* subset of $A \times B$. The *domain* of $R$ is $\{a \in A : \exists b \text{ such that } (a, b) \in R\}$. The *range* of $R$ is $\{b \in B : \exists a \in A \text{ such that } (a, b) \in R\}$.

In economics, preferences on a consumption set $X$ are formally a relation $R$ on $X \times X$. One writes $x \succeq \hat{x}$ (the decision maker weakly prefers $x$ to $\hat{x}$) iff $(x, \hat{x}) \in R$.

## 8.4 Functions

A function from a set $X$ to a set $Y$ is a relation $R$ with the following properties.

1. The domain of $R$ is $X$.
2. $R$ is single-valued: if $(x, y), (x, \hat{y}) \in R$ then $y = \hat{y}$.

If $R$ is a function then I write $f(x) = y$ to mean $(x, y) \in R$. The notation $f : X \to Y$ is read, "$f$ is a function from $X$ to $Y$." I refer to $Y$ as the target space, which may be strictly larger than the range.

Do not confuse $f$, which is the name of the function, with $f(x)$ which is the value the function takes at a point. It is wise to use different notation for function names and for values. Thus, it is better to write $y = f(x)$ than to write $y = y(x)$.

Given a set $A \subseteq X$, the *image* of $A$ is the set denoted $f(A)$ comprising all values taken by $f$ when restricted to the set $A$. Formally $f(A) = \{y \in Y : \exists x \in A \text{ such that } y = f(x)\}$. The set $f(X)$, which is the image of all of $X$, is simply the range of $f$.

Similarly, given a set $B \subseteq Y$, the *preimage* or *inverse image* of $B$ is the set, denoted, $f^{-1}(B)$, of all points in $X$ that yield values in $B$. Formally $f^{-1}(B) = \{x \in X : f(x) \in B\}$. The set $f^{-1}(B)$ is called the *preimage* or *inverse image* of $B$.

For ease of notation, when dealing with functions, if a set is a *singleton* (contains exactly one element), I often drop the brackets $\{$ and $\}$. Thus if $f(A) = \{y\}$ then I write $f(A) = y$. Similarly, if $f^{-1}(B) = \{x\}$ then I write $f^{-1}(B) = x$. And for $y \in Y$, I write $f^{-1}(y)$ rather than $f^{-1}(\{y\})$.

A function is called *one-to-one*, 1-1, *into*, or *injective* if, for any $x, \hat{x} \in X$, $f(x) = f(\hat{x})$ implies $x = \hat{x}$. If a function is 1-1 then for any $y \in Y$, $f^{-1}(y)$ is either empty or is a singleton.

A function is called *onto* or *surjective* iff $f(X) = Y$. Thus, a function is onto iff $f^{-1}(y)$ is not empty for any $y \in Y$.

A function is called a *bijection* iff it is 1-1 and onto.

The following examples may help make these definitions more concrete.

*Example* 1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^x$. Then $f$ is 1-1 but not onto, since $f(\mathbb{R}) = \mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$. (I assume that you know what $\mathbb{R}$, the set of real numbers, is even though I have not introduced it yet.) $\square$

*Example* 2. Let $f : \mathbb{R} \to \mathbb{R}_{++}$ be defined by $f(x) = e^x$. Then $f$ is 1-1 and onto. As this example suggests, any $f$ can be made onto by restricting $Y$ to $f(X)$. Thus, whether a function is onto is of interest only if we are focusing for some reason on a particular $Y$. $\square$

*Example* 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x$. Then $f$ is onto but not 1-1. For example, $f^{-1}(0) = \{-1, 0, 1\}$. $\square$

*Example* 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$. Then $f$ is a bijection, being both 1-1 and onto. $\square$

If $f$ is 1-1 and onto, a bijection, then $f^{-1}(y)$ has a single value for every $y \in Y$. In this case, I can view $f^{-1}$ as a function, called the *inverse function*, and write $f^{-1} : Y \to X$. More generally, if $f$ is 1-1 then $f^{-1}$ is a function on $f(X)$ and I can write $f^{-1} : f(X) \to X$.

### 8.5 Order.

Given a set $X$, an *order* is a relation with the following properties.

1. For every $x \in X$, $(x, x) \in R$ (reflexivity).
2. For every $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$ (antisymmetry).
3. For every $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$ (transitivity).

The order is called *total* (or *linear*) if in addition it satisfies the following property.

4. For every $x, y \in X$, either $(x, y) \in R$ or $(y, x) \in R$ (completeness).

Note that completeness implies reflexivity. The standard order on the real line $\mathbb{R}$ is a total order. On $\mathbb{R}^2$, a standard order is that $(a, b) \geq (c, d)$ iff $a \geq c$ and $b \geq d$. This is an order but it is not a total order because $(1, 2)$ and $(2, 1)$, to take an example, are not ordered.

In economics, it is common to work with *preorders*. A preorder is a relation that is reflexive and transitive, but may fail antisymmetry. In particular, preferences are given as a preorder on the consumption set, say $\mathbb{R}^2_+$. The decision maker may weakly prefer $(2,1)$ to $(1,2)$ and also vice versa (be indifferent between them) but $(2,1) \neq (1,2)$.

Given an order $R$, one can define a strict order $S$ by $(x,y) \in S$ iff $(x,y) \in R$ and $x \neq y$. The induced strict order has the following properties.

1. For every $x \in X$, $(x,x) \notin S$ (irreflexivity).
2. For every $(x,y) \in X$, if $(x,y) \in S$ then $(y,x) \notin S$ (asymmetry).
3. Transitivity.

A similar construction holds for preorders.

Let $X$ be an ordered set and let $S \subseteq X$. $x \in X$ is an upper bound for $S$ iff $(x,s) \in R$ for every $s \in S$. Let $S^*$ be the set (possibly empty) of upper bounds of $S$. The *least upper bound (LUB)* of $S$, often written $\sup S$ (sup for *supremum*), is the smallest element of $S^*$, if it exists. Similarly, the *greatest lower bound (GLB)* of $S$, often written $\inf S$ (inf for *infimum*), is the largest element, if it exists, of the set (possibly empty) of lower bounds.

Existence of a sup or inf turns out to be a critical property for many applications, including maximization. One of the key characteristics of the real numbers $\mathbb{R}$ is that it has the *Least Upper Bound Property*: Every set $S \subseteq \mathbb{R}$ that has an upper bound has a least upper bound (and similarly for lower bounds). This property does not hold for the rationals. For example, consider the set of rational numbers $r$ such that $r^2 \leq 2$. This set is bounded above (by 1.5, for example), but it has no least upper bound; it is not hard to see that if there *were* a least upper bound, the least upper bound would be $\sqrt{2}$, but it has been known since the time of the ancient Greeks that $\sqrt{2}$ is not rational.

# 9 Axiom: Infinity

## 9.1 Statement of the Axiom of Infinity

For any set $A$, define $A_+ = A \cup \{A\}$. By Pairing and Union, $A_+$ exists.

*There exists a set $S$ such that $\emptyset \in S$ and, for any $A \in S$, $A_+ \in S$.*

Any such set is called an *inductive* set. Intuitively, an inductive set must be infinite (although I have not gotten to the definition of infinity yet). Hence the Axiom of Infinity implies the existence of a particular kind of infinite set.

## 9.2 The Natural Numbers.

Identify the number 0 with $\emptyset$, 1 with $\emptyset_+ = \{\emptyset\} = \{0\}$, 2 with $\emptyset_{++} = \{\emptyset, \{\emptyset\}\} = \{0,1\}$. And so on. In general, the number $n = \{0,1,2,\ldots,n-1\}$. These are the *natural numbers*. The construction of any given natural number does *not* require the Axiom of Infinity. Nor do I need Infinity to construct any finite set of natural numbers, say $\{1, 5, 6\}$; repeated application of Pairing will suffice. What does require a new axiom, namely Infinity, is the statement that there is a set containing all of the natural numbers. Theorem 1 shows why I have to be careful about what collections I call sets.

Intuitively, there should be a set, denoted $\mathbb{N}$, comprising precisely the natural numbers. Demonstrating this turns out to be slightly delicate.

To begin, note that any inductive set contains all of the natural numbers as elements; I'm taking this as self-evident but it also follows from the construction below. One candidate approach to defining $\mathbb{N}$ is then to fix an inductive set $S$ and use Specification to identify the subset of $S$ comprising either $\emptyset = 0$ or $\emptyset_+ = 1$ or $\emptyset_{++} = 2$, and so on. If I try to formalize this, then I hit a wall because the formalization of "and so on" makes use of $\mathbb{N}$, which makes everything circular.

Alternatively, again exploiting the fact that $\mathbb{N}$ is a subset of any inductive set, one could take $\mathbb{N}$ to be the intersection of all inductive sets: $\mathbb{N}$ is the smallest inductive set. But here I encounter another problem: in order to form the intersection, one would have to start with the set of all inductive sets, and I have no assurance that such a set exists.

Fortunately, it is possible to finesse the latter problem by arguing that $\mathbb{N}$ must be the intersection of all inductive sets that are subsets of any given inductive set. Explicitly, the argument is as follows.

I start by noting that the intersection of inductive sets is an inductive set.

**Theorem 3.** *If $\mathcal{A}$ is a set of inductive sets then $\bigcap_{A\in\mathcal{A}} A$ is inductive.*

**Proof.** First, $\emptyset \in \bigcap_{A\in\mathcal{A}} A$ since $\emptyset \in A$ for every $A \in \bigcap_{A\in\mathcal{A}} A$. Second, suppose $a \in \bigcap_{A\in\mathcal{A}} A$. Then $a \in A$ for every $A \in \mathcal{A}$. Since each such $A$ is an inductive set, $a_+ \in A$ for every $A \in \mathcal{A}$. Hence $a_+ \in \bigcap_{A\in\mathcal{A}} A$, as was to be shown. ■

Fix *any* inductive set $S$ and let $T$ equal the intersection of all inductive sets, including $S$ itself, that are subsets of $S$.

**Theorem 4.** *If $A$ is an inductive set then $T \subseteq A$.*

**Proof.** By Theorem 3, $T$ is inductive, and since $A$ is inductive, $T \cap A$ is inductive. I claim that $T \cap A = T$. It is immediate, by definition of intersection, that $T \cap A \subseteq T$. Since $T \subseteq S$ (where $S$ was used to construct $T$), it follows that $T \cap A$ is an inductive set that is a subset of $S$ and hence, by definition of $T$, $T \subseteq T \cap A$. Thus $T \cap A = T$, as claimed. Finally, since (by definition of intersection) $T \cap A \subseteq A$, it follows that $T \subseteq A$, as was to be shown. ■

That is, $T$ is the unique inductive set that is a subset of every inductive set and hence would equal the intersection of all inductive sets, if I could, in fact, formally define such an intersection. Therefore, set $\mathbb{N} = T$.

### 9.3 Arithmetic

In the above construction of $\mathbb{N}$, $n_+ = n + 1$. Henceforth, I write $n + 1$ rather than $n_+$.

As should be evident from this, one can proceed recursively to define $n+2$, $n+3$, ..., and then $n + m$, and then $n \times 1$, $n \times 2$, $n \times 3$, ..., $n \times m$. And from this one can go on to definite exponentiation and so on. One can, in short, define natural number arithmetic.

There is an older, pre-Set Theory axiomatization of arithmetic called the Peano Axioms. I won't go through them in detail. An example is: if $n, m \in \mathbb{N}$ then $n = m$ iff $n + 1 = m + 1$. From our perspective, the Peano Axioms are just theorems in Set Theory.

### 9.4 Induction

One of the Peano Axioms for Arithmetic is the following theorem of Set Theory.

**Theorem 5.** *If $A$ is an inductive set and $A \subseteq \mathbb{N}$ then $A = \mathbb{N}$.*

**Proof.** By Theorem 4, $\mathbb{N} \subseteq A$. Since, by assumption $A \subseteq \mathbb{N}$, $A = \mathbb{N}$. ■

Theorem 5 is the basis for "proofs by induction" (which are just ordinary deductive proofs that happen to use this theorem).

*Example* 5. I claim that for every $n \in \mathbb{N}$, if $x_n = \sum_{i=0}^{n} i$ then $x_n = n(n + 1)/2$. To establish this, let $A = \{n \in \mathbb{N} : x_n = n/(n + 1)/2\}$. I need to show that $A = \mathbb{N}$. By Theorem 5, since $A \subseteq \mathbb{N}$, it suffices to show that $A$ is inductive. First, $0 \in A$, since $0 = \sum_{i=0}^{0} i = 0(1)/2$. Second, suppose that $n \in A$. Then $x_n = n(n + 1)/2$.

$$\begin{aligned} x_{n+1} &= \sum_{i=0}^{n+1} i \\ &= (n+1) + \sum_{i=0}^{n} i \\ &= (n+1) + \frac{n(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}, \end{aligned}$$

as was to be shown. □

### 9.5 Cardinality

Say that sets $A$ and $B$ have the same *cardinality* iff there is a bijection $f : A \to B$.

As constructed, the number $n \in \mathbb{N}$ is itself a set with $n$ elements, namely $\{0, 1, 2, \ldots, n - 1\}$. Therefore, say that a set $A$ has cardinality $n$ iff $A$ has the same cardinality as the set $\{0, 1, 2, \ldots, n-1\}$. It is not hard to show that the cardinality of a set cannot simultaneously be 3 and 11, say. A set $S$ is *finite* if its cardinality is a natural number.

Any set that is not finite is *infinite*. The set $\mathbb{N}$ itself is infinite and its cardinality is written $\aleph_0$ (I have to call it something and I have chosen to call it $\aleph_0$ instead of, say, Alice). A set is *countably infinite* if there is a bijection from the set to $\mathbb{N}$. In this case, the set's cardinality is $\aleph_0$, the same as that of $\mathbb{N}$.

This definition of cardinality has some consequences that you may find counterintuitive. One of these is that there are proper subsets of $\mathbb{N}$, for example, $S = \{1, 3, 5, 7, \ldots\}$, that also have cardinality $\aleph_0$. If you take elements away from a finite set, then that set's cardinality always goes down. This is not true for infinite sets.

The following result states that a countable union of countable sets is countable. One implication is that the set of rational numbers is countable, even though the set of rational numbers strictly contains $\mathbb{N}$.

**Theorem 6.** *Let $\mathcal{A}$ be countable and suppose that $A$ is countable for every $A \in \mathcal{A}$. Then*

$$\bigcup_{A \in \mathcal{A}} A$$

*is countable.*

**Proof.** Since $\mathcal{A}$ is countable it can be written in the form $\{A_0, A_1, \ldots\}$. For each $n \in \mathbb{N}$, since $A_n$ is countable, it can be written in the form $\{a_{n0}, a_{n1}, \ldots\}$. Informally, imaging arranging the elements of the union into a giant (infinite) matrix, with $A_n$ occupying line $n$. Then construct a set by choosing the element $a_{00}$ in the upper left and after that zigzagging: $\{a_{00}, a_{01}, a_{10}, a_{20}, a_{11}, a_{02}, a_{03}, a_{12}, \ldots\}$ (drawing a picture may help). It is easy to verify that this set, which is countable, equals the desired union. ■

Despite this result, there are sets that are so much larger than $\mathbb{N}$ that they are not countably infinite. One such set is the set of real numbers, $\mathbb{R}$. I provide a proof of this later in the course. Any such set is *uncountable*.

The cardinality or $\mathbb{R}$ is often written $2^{\aleph_0}$. This notation comes from the following fact. Consider first a finite set, say $S = \{a, b, c\}$. The set of all subsets of $S$ (the power set of $S$), is

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\},$$

which has $8 = 2^3$ elements. In general, if the cardinality of a finite set is $n$ then the cardinal of the power set is $2^n$. As the notation $2^{\aleph_0}$ suggests, one can show that the cardinality of $\mathbb{R}$ equals the cardinality of the power set of $\mathbb{N}$.

# 10 Axiom: Choice

## 10.1 Statement of the Axiom of Choice.

> *Let $\mathcal{A}$ be a non-empty set of non-empty sets. Let $C = \bigcup_{A \in \mathcal{A}}$. There exists a function $f : \mathcal{A} \to C$ such that for each $A \in \mathcal{A}$, $f(A) \in A$.*

That is, the function $f$ chooses one element from each set in $\mathcal{A}$, hence the name Axiom of Choice.

If $\mathcal{A}$ contains a single set, say $A$, then Choice is clearly satisfied: since $A$ is not empty, there is an $a \in A$; set $f(A) = a$. By induction, Choice holds for any (non-empty) finite set of (non-empty) sets. This is true even if the $A \in \mathcal{A}$ are themselves infinite.

While induction, as expressed in Theorem 5, allows one to construct choice functions for any set $\mathcal{A}$ of $n \in \mathbb{N}$ sets, induction does not imply that one can construct a choice function for an infinite set of sets. A new axiom, Choice, is needed to guarantee the existence of a choice function in this case. The situation is somewhat analogous to the situation with the construction of $\mathbb{N}$. Recall that I could define every $n \in \mathbb{N}$ without invoking the Axiom of Infinity, but I required a new axiom to be able to call the infinite collection $\mathbb{N}$ a set.

There was a time when Choice was so contentious that it had a special status in Set Theory. The reason is that Choice implies the existence of some highly non-intuitive, ill-behaved sets. For example, Choice gives rise to the Banach-Tarski paradox, which says, roughly, that there is a way to cut a sphere up into a finite number of pieces that can be reassembled (via shifting and rotating; no stretching or bending) into two spheres of the same volume as the original sphere. On the other hand, Choice plays an important positive role in some of the mathematics used in economics. One application is to the existence of preference preorders that rationalize choice structures obeying the Strong Axiom of Revealed Preference.

## 10.2 Cartesian Products II.

The definition of Cartesian product given in Section 8.2 was for the product of two sets. This construction can easily be extended by induction to any finite number of sets. But the construction does not extend to products of an infinite number of sets, because, in that construction, it is not clear what set the infinite product set is an element of.

There is, fortunately, another approach that scales easily up to infinite products. Consider again the product of two sets $A_1$ and $A_2$. Rather than define $A_1 \times A_2$ as in Section 8.2, do the following. Let $C = A_1 \cup A_2$ and consider any function $f : \{1, 2\} \to C$ such that $f(1) \in A_1$ and $f(2) \in A_2$. Any such function is equivalent to a point in $A_1 \times A_2$: if $f(1) = a_1$ and $f(2) = a_2$ then the point is $(a_1, a_2)$.

Then $A_1 \times A_2$ can be identified with the set of such functions. Formally, a function $f : \{1, 2\} \to C$ is a subset of $\{1, 2\} \times C$. Therefore, $A_1 \times A_2$ can be identified with the set

$$\{f \in \mathbb{P}(\{1,2\} \times C) : f : \{1,2\} \to C \text{ and } \forall i \in \{1,2\}, f(i) \in A_i\}.$$

To generalize this, let $\mathcal{A}$ be a set of sets, let $I$ be another set, and let $g : I \to \mathcal{A}$. The interpretation is that $I$ indexes the sets in $\mathcal{A}$. In the example above, $I = \{1,2\}$, $g(1) = A_1$ and $g(2) = A_2$. If $I = \mathbb{N}$ then I typically write $\mathcal{A} = \{A_0, A_1, \dots, \}$. In the general case, I do not require that $g$ be either 1-1 or onto: the same set can be indexed more than once (this is how I would construct $\mathbb{R}^\infty$) and some sets may not be indexed at all.

Fix $I$, $\mathcal{A}$, and $g$. I construct the Cartesian product as follows. Let $C = \bigcup_{A \in \mathcal{A}} A$. Then the Cartesian product is

$$\{f \in \mathbb{P}(I \times C) : f : I \to C \text{ and } \forall i \in I, f(i) \in g(i)\}.$$

Note that this definition makes use of $I \times C$. So this definition of Cartesian product requires the use of the definition of pairwise Cartesian products given in Section 8.2.

A subtlety with infinite Cartesian products is that the Axiom of Choice is sometimes needed to guarantee that the product is not empty. In particular, note that an element of the product is a choice function. Choice is not, however, needed to establish non-emptiness of $\mathbb{R}^\infty$.

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