Econ 511
Professor: John Nachbar
Friday 20, 2012

The Rational and Real Numbers

1 Overview.

Having constructed the natural numbers $\mathbb{N}$ out of nothing (more accurately, out of $\emptyset$ and sets containing $\emptyset$), one can construct the integers $\mathbb{Z}$ out of $\mathbb{N}$, the rational numbers $\mathbb{Q}$ out of $\mathbb{Z}$, and the real numbers $\mathbb{R}$ out of $\mathbb{Q}$.

The reason for doing this carefully is that $\mathbb{R}$ turns out to have properties that confound everyday intuition. For example, consider a circle of unit circumference. This can be identified with the interval $[0, 1) \subseteq \mathbb{R}$. The Axiom of Choice implies that I can divide this circle into a countably infinite number of pieces called that are identical and disjoint. Each piece is a complicated set of points; not a simple arc. If I try to assign a generalized notion of length to these pieces then I confront the following problem. Since the pieces are identical, they should all have the same length. But if the length of each piece is, say, $\varepsilon > 0$, and each piece has the same length, then the sum of any $n > 1/\varepsilon$ of the pieces is greater than the overall circumference of the circle, namely 1. On the other hand, If the length of each piece is zero then the sum of all the lengths is zero. Lebesgue measure, which is a generalized notion of ordinary length, deals with this problem by *not* assigning length to such sets.

The set theoretic construction of $\mathbb{R}$ establishes that $\mathbb{R}$, although weird in a number of ways when examined carefully, is nevertheless well defined. These notes sketch the construction of $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ and record some basic properties of these sets.

2 The Rational Numbers.

I construct the integers out of ordered pairs (n_1, n_2) of natural numbers. Intuitively, (n_1, n_2) is the integer equal to $n_1 - n_2$, although the formal construction uses negative integers to define subtraction rather than the other way around. Note that the integer -1 represents both $(2, 3)$ and $(4, 5)$, since $2 - 3 = 4 - 5 = -1$. To deal with this potential ambiguity, call (n_1, n_2) and $(\hat{n}_1, \hat{n}_2)$ *equivalent* if $n_1 + \hat{n}_2 = \hat{n}_1 + n_2$ (i.e., $n_1 - n_2 = \hat{n}_1 - \hat{n_2}$). The set of ordered pairs equivalent to (n_1, n_2) is called an *equivalence class* and is written $[(n_1, n_2)]$. If (n_1, n_2) and $(\hat{n}_1, \hat{n}_2)$ are equivalent then $[(n_1, n_2)] = [(\hat{n}_1, \hat{n}_2)]$. Each equivalence class "is" an integer. Thus, for example, the integer conventionally written -1 represents the equivalence class $[(0, 1)] = \{(0, 1), (1, 2), \dots\}$. The set of integers, conventionally written $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, is, as defined here, a subset of $\mathbb{P}(\mathbb{N} \times \mathbb{N})$.

I can then define arithmetic for $\mathbb{Z}$ as follows. Given two integers z and $\hat{z}$, suppose z is $[(n_1, n_2)]$ and $\hat{z}$ is $[(\hat{n}_1, \hat{n}_2)]$. Define $z + \hat{z}$ to be the integer $[(n_1 + \hat{n}_1, n_2 + \hat{n}_2)]$; intuitively, $(n_1 - n_2) + (\hat{n}_1 - \hat{n}_2) = (n_1 + \hat{n}_1) - (n_2 + \hat{n}_2)$. For example, 3 is $[(3, 0)]$ and -2 is $[(0, 2)]$, hence $3 + (-2)$ is $[(3 + 0, 0 + 2)] = [(3, 2)]$, which is the integer 1; $3 + (-2) = 1$. Note that the calculation does not depend on whether, for example, I write -2 as $[(0, 2)]$ or $[(5, 7)]$.

For subtraction, define $z - \hat{z}$ to be $z + (-\hat{z})$, where if $\hat{z}$ is $[(\hat{n}_1, \hat{n}_2)]$ then $-\hat{z}$ is $[(\hat{n}_2, \hat{n}_1)]$. For example, to compute $2 - 3$, since 2 is $[(2, 0)]$ and 3 is $[(3, 0)]$, -3 is $[(0, 3)]$ and $2 - 3$ is $2 + (-3)$, which is $[(2 + 0, 0 + 3)] = [(2, 3)]$, which is the integer -1; $2 - 3 = -1$.

Finally, for multiplication, define $z\hat{z}$ to be $[(n_1\hat{n}_1 + n_2\hat{n}_2, n_1\hat{n}_2 + n_2\hat{n}_1)]$; intuitively, $(n_1 - n_2)(\hat{n}_1 - \hat{n}_2) = (n_1\hat{n}_1 + n_2\hat{n}_2) - (n_1\hat{n}_2 + n_2\hat{n}_1)$. For example, to compute the product of -2 and -3, -2 is $[(0, 2)]$ and -3 is $(0, 3)$, and so the product of -2 and -3 is given by $[(0 + 6, 0 + 0)] = [(6, 0)]$, or 6: $(-2) \times (-3) = 6$.

From $\mathbb{Z}$ one can then construct the rational numbers out of pairs of integers (z_1, z_2), $z_2 \neq 0$. Intuitively, (z_1, z_2) is the rational number equal to z_1/z_2, although the formal construction uses rational numbers to define division rather than the other way around. Say that (z_1, z_2) and $(\hat{z}_1, \hat{z}_2)$ are in the same equivalence class if $z_1\hat{z}_2 = \hat{z}_1 z_2$ (i.e., $z_1/z_2 = \hat{z}_1/\hat{z}_2$, hence $z_1\hat{z}_2 = \hat{z}_1 z_2$). Each such equivalence class of integer pairs "is" a rational number. For example, the integer conventionally written $1/2$ represents the equivalence class $[(1, 2)] = [(1, 2), (-1, -2), (2, 4), \dots]$. The set of rational numbers, conventionally written $\mathbb{Q}$ is, as defined here, a subset of $\mathbb{P}(\mathbb{Z} \times \mathbb{Z})$.

One can then go on to define addition, subtraction, multiplication, and also division for rational numbers. The construction is entirely analogous to that for the arithmetic operations on $\mathbb{Z}$, so I omit it.

3 The Real Field.

The rational numbers, together with addition and multiplication, can be shown to form an algebraic object called a *field*. A definition can be found in Rudin (1976). Briefly, the fact that the rationals form a field means that it has familiar properties such as $p + q = q + p$ (addition is commutative) and $p(q + r) = pq + pr$ (addition and multiplication are distributive). In addition, $\mathbb{Q}$ has an additive identity, namely 0, and every $p \in \mathbb{Q}$ has an additive inverse, namely $-p$. And $\mathbb{Q}$ has a multiplicative identity, namely 1, and if $p \neq 0$, then it also has a multiplicative inverse, namely $1/p$. With the standard order $\geq$, $\mathbb{Q}$ is an *ordered field*.

Because $\mathbb{Q}$ is an ordered field, it is, in many ways, an extremely nice space in which to work. It has, however, a major drawback for certain applications: it has "holes." One concise and elementary way to express this is that $\mathbb{Q}$ does not satisfy the Least Upper Bound (LUB) property: there are subsets of $\mathbb{Q}$ that are bounded above but that have no least upper bound; the missing least upper bounds are holes in $\mathbb{Q}$. For example, it follows from the fact that $\sqrt{2}$ is not rational (this requires proof) and Theorem 4 below that $\{x \in \mathbb{Q} : x^2 \leq 2\}$ has no least upper bound, even though it is bounded above.

Theorem 1 below states that there exists an ordered field with the LUB property, namely the reals, $\mathbb{R}$. In the statement of the theorem, two ordered fields are *isomorphic* iff there is a bijection between their elements that preserves addition, multiplication and order: if h is the bijection then for any x, y in the ordered field, $h(x+y) = h(x)+h(y)$, $h(xy) = h(x)h(y)$, and $y > x$ iff $h(y) > h(x)$. If two ordered fields are isomorphic then they are really the *same* ordered field, but with different names for the elements. Think about discussing $\mathbb{Q}$ in some language other than English; $\mathbb{Q}$ is still $\mathbb{Q}$, even though some of the words are different. The statement that $\mathbb{R}$ is unique up to isomorphism means that $\mathbb{R}$ is the *only* ordered field with the LUB property; any other ordered field with the LUB property is just $\mathbb{R}$ in another language. Finally, the statement that $\mathbb{R}$ contains $\mathbb{Q}$ as an ordered subfield merely means that $\mathbb{Q}$ is a subset of $\mathbb{R}$ and that the arithmetic operations and order on $\mathbb{R}$ are, when restricted to $\mathbb{Q}$, the standard arithmetic operations and order on $\mathbb{Q}$.

Theorem 1. *There exists an ordered field that satisfies the LUB property. This field is unique up to isomorphism. Call this, essentially unique, field* $\mathbb{R}$. $\mathbb{R}$ *contains* $\mathbb{Q}$ *as an ordered subfield.*

Proof. The construction of $\mathbb{R}$ can be found in Rudin (1976). The proof of the isomorphism claim can be found in the proof of Theorem 5.34 in Hewitt and Stromberg (1965).

Here is a brief outline of the construction of $\mathbb{R}$. Recall that the natural number 3 is defined to be the set $\{0, 1, 2\}$. More generally, the natural number n is defined to be the set of natural numbers less than n. In a similar way, one can identify a rational number $x \in \mathbb{Q}$ with the set of rational numbers less than x: $C_x = \{r \in \mathbb{Q} : r < x\}$. The latter set is called a *cut*.

One can define addition, multiplication, and order for these cuts. Addition is set addition: if $x, y \in \mathbb{Q}$, then $C_x + C_y = \{r \in \mathbb{Q} : \exists\, p \in C_x, q \in C_y \text{ such that } r = p+q\}$. One can show that $C_x + C_y$ is indeed a cut and that $C_x + C_y = \{r \in \mathbb{Q} : r < x + y\}$. The definition of multiplication is more finicky because one has to take into account the multiplication of negative rationals; I'll spare you the details. Having defined multiplication for cuts, however, one can show that $C_x C_y = \{r \in \mathbb{Q} : r < xy\}$. Finally, for order, if $x, y \in \mathbb{Q}$, then $C_x \leq C_y$ iff $C_x \subseteq C_y$. With addition, multiplication, and order thus defined, one can show that these cuts constitute an ordered field that is isomorphic to $\mathbb{Q}$: for any $x \in \mathbb{Q}$, C_x is just another name for x.

The idea in the construction of $\mathbb{R}$ is to identify the real number $x \in \mathbb{R}$ with the cut $\{q \in \mathbb{Q} : q < x\}$. But this is circular, since $x \in \mathbb{R}$ and $\mathbb{R}$ has not yet been defined. To avoid circularity, define a general cut as follows. A cut is a set $C \subseteq \mathbb{Q}$ such that (i) $C \neq \emptyset$, (ii) C is bounded above: there is a $y \in \mathbb{Q}$ such that for every $\gamma \in C$, $\gamma \leq y$, (iii) if $q \in C$, $\gamma \in \mathbb{Q}$, and $\gamma < q$ then $\gamma \in C$, and (iv) if $q \in C$ then there is an element $\gamma \in C$ such that $\gamma > q$. For any $x \in \mathbb{Q}$, the cut C_x as defined above satisfies all four properties.

The set $\mathbb{R}$ is the set of all cuts. Defining addition, multiplication, and order as above, $\mathbb{R}$ is an ordered field. Moreover, the LUB property holds: for a set of cuts $\mathfrak{C}$ that is bounded above (there is a cut B such that $C \subseteq B$ for every $C \in \mathfrak{C}$), the least upper bound is the set $\cup_{C \in \mathfrak{C}} C$, which is indeed a cut.

The additive identity for $\mathbb{R}$ is easily seen to be $C_0 = \{q \in \mathbb{Q} : q < 0\}$, which, of course, we simply call 0. Similarly, the multiplicative identity for $\mathbb{R}$ can be shown to be $C_1 = \{q \in \mathbb{Q} : q < 1\}$, which, of course, we simply call 1. One can then show by induction arguments that for any $x \in \mathbb{Q}$, the cut for x is indeed $C_x = \{q \in \mathbb{Q} : q < x\}$. Thus $\mathbb{R}$ contains (an ordered field isomorphic to) $\mathbb{Q}$ as an ordered subfield. ■

For future reference, note that by property (iv) of a cut, if a rational number q is in the cut defining x, then $x > q$. And, by a similar argument, if a rational number r is not in the cut for x then $r \geq x$.

The construction of $\mathbb{R}$ via cuts is one of three common constructions. Each of these constructions have their own strengths and weaknesses. The cut construction is admittedly abstract and the required definition of multiplication is inelegant. But the cut construction is similar to the original construction of $\mathbb{N}$, it gives a unique representation of each real number, and easily generates the LUB property, as well as Theorems 3 and 4 below. An alternative is to define elements of $\mathbb{R}$ in terms of infinite decimal expansions, which has the great virtue of being familiar, but in many respects it is difficult to work with at a formal level. And it does not give reals unique representations (since $1.00\cdots = 0.99\ldots$). Finally, one can define reals using equivalence classes of Cauchy sequences. This approach gives completeness of $\mathbb{R}$ by construction. But it requires first defining Cauchy sequences, some of the details are again messy, and again it does not give reals unique representations (since there are infinitely many sequences converging to the same real number).

4 Basic Properties of $\mathbb{R}$.

Theorem 2. *$\mathbb{Q}$ is countable. $\mathbb{R}$ is uncountable.*

Proof. Countability of $\mathbb{Q}$ follows from an argument similar to that used to show that countable unions of countable sets are countable. I postpone the proof that $\mathbb{R}$ is *not* countable to later in the course. ■

One corollary of Theorem 1 and 2 is that $\mathbb{Q}$ does not have the LUB property. Theorem 1 implies that any ordered field with the LUB property is isomorphic to $\mathbb{R}$. And Theorem 2 says that $\mathbb{Q}$ is not isomorphic to $\mathbb{R}$ because it has a different cardinality.

Remark 1. As is hinted at by the construction of $\mathbb{R}$ out of subsets of $\mathbb{Q}$, one can show that the cardinality of $\mathbb{R}$ is the same as the cardinality of the set of subsets of

$\mathbb{Q}$ (or $\mathbb{N}$), which is usually called $2^{\aleph_0}$, where $\aleph_0 = |\mathbb{N}|$. □

The following properties are easily established for $\mathbb{Q}$.

Theorem 3 (Archimedean Property). *For any $x, y \in \mathbb{R}$, if $x > 0$ then there exists $n \in \mathbb{N}$ s.t. $nx > y$.*

Proof. The proof follows from the construction of $\mathbb{R}$. Take it as given that the Archimedean Property holds for $\mathbb{Q}$; this is not hard to establish. The cut for y is bounded above and hence there is a rational number $r > y$. Since $x > 0$, there is a positive rational number p in the cut for x. Since p is in the cut for x, $x > p$. Since the Archimedean property holds for $\mathbb{Q}$, take a natural number $n > 0$ such that $np > r$. Then $nx > np > r > y$. (Statements such as, "if $x > p$ and $n > 0$, then $nx > np$," follow from the fact that $\mathbb{R}$ is a field.) ■

Theorem 4 (Betweenness). *For any $x, y \in \mathbb{R}$ s.t. $y > x$, there exists $p \in \mathbb{Q}$ s.t. $y > p > x$.*

Proof. Almost immediate from the construction of $\mathbb{R}$. If $y > x$ then the cut defining y contains a rational number r that is not in the cut defining x. By the definition of a cut, there is also a rational number $p > r$ in the cut for y. Then $y > p > r \geq x$. ■

Remark 2. Betweeness may seem to imply that the real and rational numbers should have the same cardinality, at least approximately. Theorem 2 says that this intuition is wrong. If you find this disturbing then welcome to the world of $\mathbb{R}$. □

5 What exactly is in $\mathbb{R}$?

My treatment here will be informal. One can show that the elements of $\mathbb{R}$ can be identified with the set of all decimal expansions. This means that in one sense the elements of $\mathbb{R}$ are all familiar: they are close in some appropriate sense to numbers such as 2.718 and 3.14. The problems with $\mathbb{R}$ come from the fact that to pin down a real number *exactly* one needs the *entire* infinite decimal expansion, and this object can be unbelievable complicated.

One way to understand what is in $\mathbb{R}$ is the following. Consider the set of numbers, call it $\mathbb{G}$, that have the property that they can be computed to arbitrary accuracy by an idealized version of a computer (a Turing machine or equivalent) that has infinite memory but only a finite program. The elements of $\mathbb{G}$ are the *computable* numbers. For a formal definition of computable number, see, for example, Cutland (1980). A number that is *not* in $\mathbb{G}$ is, in an important sense, purely abstract. We have no way of communicating it precisely to anyone else. We might be able to say that it lies in the interval, say $(0.9192, 0.9193)$, but then so do an uncountable infinity of other real numbers.

What is in $\mathbb{G}$? $\mathbb{N} \subseteq \mathbb{G}$ since the "finite program" for n can be just n itself. $\mathbb{Z} \subseteq \mathbb{G}$ since, as discussed above, the elements of $\mathbb{Z}$ can be represented as pairs of natural numbers. Similarly, $\mathbb{Q} \subseteq \mathbb{G}$.

Less obviously, $\mathbb{G}$ contains all *algebraic* real numbers, which are the real solutions (if any exist) to expressions of the form,

$$a_0 + a_1 x + \cdots + a_N x^N = 0,$$

where the a_n are integers. Any rational number is algebraic; for example, $3/4$ is the solution to $-3 + 4x = 0$. The standard example of an irrational algebraic number is $\sqrt{2}$, which is the solution to $-2 + x^2 = 0$. Any such number can computed to arbitrary precision by a computer, the only constraints being memory and time. A software package such as *Mathematica* or *Maple* will actually do this calculation.

Even less obviously, $\mathbb{G}$ contains numbers like π and e that are not algebraic. Again, the fact that π is computable should be familiar from the fact that a software package such as *Mathematica* can, in fact, compute π to arbitrary precision.

$\mathbb{G}$ contains every real number you have ever actually used for anything. But $\mathbb{G}$ is *countable*, because the set of finite programs is countable. This means that as large as $\mathbb{G}$ is, $\mathbb{R}$, being uncountable, is vastly, unimaginably larger. For example, under the uniform distribution, the probability of picking an element of $\mathbb{G}$ from the $[0, 1]$ interval is *zero*: with probability one, you will pick a number that has no finite description.

6 Basic facts about $\mathbb{R}^N$.

$\mathbb{R}^N$ denotes the N-fold cartesian product of $\mathbb{R}$. Thus $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

A point $x \in \mathbb{R}^N$, $x = (x_1, \ldots, x_N)$, is called a *vector*. It is sometimes useful to interpret a vector x as a *directed line segment*, an arrow with a base at 0 and its head at x. But this is merely one interpretation.

Vector addition: if $x, y \in \mathbb{R}$ then

$$x + y = (x_1 + y_1, \ldots, x_N + y_N).$$

Scalar multiplication: if $a \in \mathbb{R}$ and $x \in \mathbb{R}^N$ then

$$ax = (ax_1, \ldots, ax_N),$$

The point $a \in \mathbb{R}$ is called a *scalar*.

Abusing notation, I let 0 represent both the scalar zero and the vector $(0, \ldots, 0)$. To avoid potential ambiguity, some texts use instead $\theta = (0, \ldots, 0)$. But the interpretation of 0 should be clear from context.

Definition 1. *If $x, y \in \mathbb{R}^N$,*

$$x \cdot y = \sum_{n=1}^{N} x_n y_n.$$

Definition 2. *If $x \in \mathbb{R}^N$,*

$$\|x\| = (x \cdot x)^{1/2}.$$

If $n = 1$ then $\|x\| = |x|$, standard absolute value.

Theorem 5. *Consider any $a \in \mathbb{R}$ and $x \in \mathbb{R}^N$.*

1. $\|x\| \geq 0$.
2. $\|ax\| = |a|\|x\|$.
3. $\|x\| = 0$ *iff* $x = 0$.

Proof. Omitted but almost immediate from the definition. ∎

Theorem 6 (Schwartz Inequality)**.** *If $x, y \in \mathbb{R}^N$ then*

$$|x \cdot y| \leq \|x\| \; \|y\|$$

Proof. If $x \cdot y = 0$ then we are done, since $\|x\|, \|y\| \geq 0$. Therefore, assume $x \cdot y \neq 0$.

For any $\lambda \in \mathbb{R}$, since a sum of squares is always non-negative,

$$\begin{aligned} 0 &\leq (x - \lambda y) \cdot (x - \lambda y) \\ &= \|x\|^2 - 2\lambda(x \cdot y) + \lambda^2 \|y\|^2. \end{aligned}$$

Set

$$\lambda = \frac{\|x\|^2}{x \cdot y},$$

which is well defined since $x \cdot y \neq 0$. Therefore,

$$0 \leq \|x\|^2 - 2\frac{\|x\|^2}{(x \cdot y)}(x \cdot y) + \frac{\|x\|^4}{(x \cdot y)^2}\|y\|^2.$$

Collecting terms and rearranging yields,

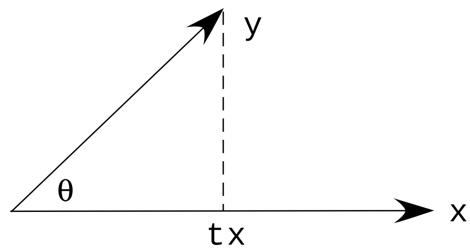
$$\|x\|^2 (x \cdot y)^2 \leq \|x\|^4 \|y\|^2.$$

Since $\|x\| > 0$ (if $\|x\| = 0$ then $x = 0$, hence $x \cdot y = 0$), one can divide both sides by $\|x\|^2$ to get

$$(x \cdot y)^2 \leq \|x\|^2 \|y\|^2.$$

Taking the square root of both sides yields the result. ∎

Remark 3. Label x and y so that $\theta \geq 0$. Assume that θ lies strictly between 0 and 90 degrees. Let t be the scalar such that the line segment from 0 to tx is at right angles to the line segment from tx to y.



Then the points 0, tx, and y form a right triangle, with hypotenuse given by the line segment from 0 to y. By the Pythagorean Theorem, $\|y\|^2 = \|tx\|^2 + \|y - tx\|^2$. Hence $y \cdot y = t^2(x \cdot x) + (y - tx) \cdot (y - tx) = t^2(x \cdot x) + y \cdot y - 2t(x \cdot y) + t^2(x \cdot x)$. Cancelling the $y \cdot y$ and collecting terms yields

$$t = \frac{x \cdot y}{x \cdot x}.$$

By the assumption that θ lies strictly between 0 and 90 degrees, t is positive.

By the definition of cosine (length of the side adjacent to the angle divided by the length of the hypotenuse):

$$\cos(\theta) = \frac{\|tx\|}{\|y\|} = \frac{x \cdot y}{x \cdot x} \frac{(x \cdot x)^{\frac{1}{2}}}{\|y\|} = \frac{x \cdot y}{\|x\|\|y\|}.$$

Rewriting,

$$x \cdot y = \|x\|\|y\| \cos(\theta).$$

This implies the Schwartz inequality. Specifically, for θ between 0 and 90 degrees, $\cos(\theta) \geq 0$ and hence $x \cdot y \geq 0$. Since $\cos(\theta) \leq 1$, $0 \leq x \cdot y \leq \|x\|\|y\|$. One can extend this argument to θ in other ranges. One merit of the formal proof of the Schwartz inequality is that it is "legal;" it does not depend on machinery, like trigonometry, that we have not yet developed.

The trigonometric argument actually gives a stronger conclusion: $|x \cdot y| \leq \|x\|\|y\|$ with $|x \cdot y| = \|x\|\|y\|$ if and only if θ equals 0 degrees. That is, $|x \cdot y| = \|x\|\|y\|$ if and only if x and y are *collinear*, meaning that these vectors, interpreted as directed line segments, point in either the same direction or in exactly opposite directions.

Lastly, note that if θ is 90 degrees then $\cos(\theta) = 0$. Hence $x \cdot y = 0$ iff x and y, considered as directed line segments, are orthogonal (at right angles to each other). □

Theorem 7 (Triangle Inequality)**.** *If $x, y \in \mathbb{R}^N$ then*

$$\|x + y\| \leq \|x\| + \|y\|.$$

Proof. By the Schwartz Inequality, $x \cdot y \leq |x \cdot y| \leq \|x\|\ \|y\|$. Therefore,

$$\begin{aligned}\|x+y\|^2 &= (x+y)\cdot(x+y)\\ &= x\cdot x + 2x\cdot y + y\cdot y\\ &\leq \|x\|^2 + 2\|x\|\ \|y\| + \|y\|^2\\ &= (\|x\|+\|y\|)^2.\end{aligned}$$

Taking the square root of both sides yields the result. ■

Remark 4. Consider vectors x and y and suppose that x and y are *not* collinear. Then the points 0, x, and $x+y$ form a triangle with sides given by the segment from 0 to x, the segment from x to $x+y$ (which has length $\|y\|$), and the segment from 0 to $x+y$. A basic fact about triangles in $\mathbb{R}^N$ is that if one takes any two sides, the sum of those lengths exceeds the the length of the third. So, in particular, $\|x\|+\|y\| > \|x+y\|$. On the other hand, if x and y are collinear then the triangle collapses and $\|x\|+\|y\| = \|x+y\|$. Theorem 7 records a (slightly weak) version of this fact, without any reliance on facts about triangles in $\mathbb{R}^N$. □

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