Econ 4111
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Quasi-concavity

1 Definitions and Basic Facts.

Definition 1. *Let $C \subseteq \mathbb{R}^N$ be convex. Consider $f : C \to \mathbb{R}$.*

1. *The function f is* quasi-concave *iff for any $x, \hat{x} \in C$, if $f(x) \geq f(\hat{x})$ then for any $\theta \in (0,1)$, setting $x_\theta = \theta x + (1-\theta)\hat{x}$, $f(x_\theta) \geq f(\hat{x})$.*

2. *The function f is* strictly quasi-concave *iff for any $x, \hat{x} \in C$, if $x \neq \hat{x}$ and $f(x) \geq f(\hat{x})$ then for any $\theta \in (0,1)$, setting $x_\theta = \theta x + (1-\theta)\hat{x}$, $f(x_\theta) > f(\hat{x})$.*

The function f is quasi-convex *iff $-f$ is quasi-concave. It is* strictly quasi-convex *iff $-f$ is strictly quasi-concave.*

Geometrically this says the following. Given a point $y \in \mathbb{R}$, the *upper contour set* of f is the set

$$\{x \in C : f(x) \geq y\}.$$

Thus, the upper contour set is the set of points that get values at least as large as y. For some values of y, this set may be empty. The function f is quasi-concave iff every upper contour set is convex. Conversely, f is quasi-convex iff its lower contour sets are convex, where the lower contour set is

$$\{x \in C : f(x) \leq y\}.$$

Theorem 1. *Let $C \subseteq \mathbb{R}^N$ be convex. If $f : C \to \mathbb{R}$ is concave then f is quasi-concave. The converse is false.*

Proof. If f is concave and $f(x) \geq f(\hat{x})$ then, by concavity, for any θ, $f(x_\theta) \geq \theta f(x) + (1-\theta)f(\hat{x}) \geq \theta f(\hat{x}) + (1-\theta)f(\hat{x}) = f(\hat{x})$.

To show that the converse is false, it suffices to provide an example. ■

Quasi-concave functions need not be concave, however. For example, if the domain is a convex subset of $\mathbb{R}$ then any monotone function is quasi-concave.

Example 1. $f(x) = e^x$ is quasi-concave even though it is *strictly convex.* For any $y > 0$, $U(y) = [\ln(y), \infty)$, which is convex. For $y \leq 0$, $U(y) = \mathbb{R}$, which is also convex. □

Example 2. $f(x) = x^2$ is not quasi-concave. For any $y > 0$, $U(y) = (-\infty, -\sqrt{y}] \cup [\sqrt{y}, \infty)$, which is not convex. □

Here is a quasi-concave function that is neither concave nor monotone.

Example 3. $f(x) = e^{-x^2}$ is quasi-concave (the function is similar to the standard normal density). For any $y \in (0, 1]$, $U(y) = [-\sqrt{\ln(1/y)}, \sqrt{\ln(1/y)}]$. If $y \leq 0$ then $U(y) = \mathbb{R}$. If $y > 1$ then $U(y) = \emptyset$. All of these sets are convex. □

Finally, here is an example of a quasi-concave, but not concave, function defined on a subset of $\mathbb{R}^2$.

Example 4. Define $f : \mathbb{R}^2_+ \to \mathbb{R}$ by $f(x) = x_1 x_2$. This function is quasi-concave. □

2 Quasi-concavity and Differentiation.

Recall that a $\mathcal{C}^2$ function f is concave iff $D^2 f(x)$ is negative semi-definite for all $x \in C$; if $D^2 f(x)$ is negative definite for all $x \in C$ then f is strictly concave. There is a similar characterization for quasi-concavity. The twist is that while concavity requires that $v' D^2 f(x) v$ be negative for all v, quasi-concavity requires only that $v' D^2 f(x) v$ be negative for v such that $\nabla f(x) \cdot v = 0$. Geometrically, this says that $D^2 f(x)$ needs to be negative definite on the tangent to the level set of f through x. If $\nabla f(x) = 0$ then the tangent is all of $\mathbb{R}^N$ and the condition is equivalent to checking concavity.

Theorem 2. *Let $C \subseteq \mathbb{R}^N$ be convex and let $f : C \to \mathbb{R}$ be twice differentiable. Let $T_{\nabla f(x)} = \{v \in \mathbb{R}^n : \nabla f(x) \cdot v = 0\}$.*

1. *Let f be $\mathcal{C}^2$. Fix $x \in C$. If $D^2 f(x)$ is negative definite on $T_{\nabla f(x)}$, then f is locally strictly quasi-concave.*

2. *For every $x \in C$, if $D^2 f(x)$ is negative semi-definite on $T_{\nabla f(x)}$, then f is quasi-concave.*

3. *If f is quasi-concave and $\mathcal{C}^2$ then, for every $x \in C$, $D^2 f(x)$ is negative semi-definite on $T_{\nabla f(x)}$.*

Proof. Omitted. ■

If $D^2 f(x)$ is negative definite on $T_{\nabla f(x)}$ for all $x \in C$ then f is *differentiably strictly quasi-concave.*

If $\nabla f(x) \neq 0$, differentiable strict quasi-concavity can be checked by checking the determinants of certain submatrices of the the *bordered Hessian*:

$$\begin{bmatrix} D^2 f(x) & \nabla f(x) \\ \nabla f(x)' & 0 \end{bmatrix}.$$

For our purposes, the most important aspect of this is that the bordered Hessian itself has a non-zero determinant.

Theorem 3. *If f is differentiably strictly quasi-concave and $\nabla f(x) \neq 0$ then*

$$\left| \begin{array}{cc} D^2 f(x) & \nabla f(x) \\ \nabla f(x)' & 0 \end{array} \right| \neq 0.$$

Proof. To simplify notation, let

$$A = \left[\begin{array}{cc} D^2 f(x) & \nabla f(x) \\ \nabla f(x)' & 0 \end{array} \right].$$

Since A is $(N+1) \times (N+1)$, it suffices to show that the column space of A is $N+1$-dimensional.

For any $v \in T_{\nabla f(x)}$,

$$A \left[\begin{array}{c} v \\ 0 \end{array} \right] = \left[\begin{array}{c} D^2 f(x) v \\ 0 \end{array} \right].$$

Since $T_{\nabla f(x)}$ is an $N-1$ dimensional linear subspace (plane through the origin), and since $D^2 f(x) v \neq 0$ for any $v \in T_{\nabla f(x)}$, $v \neq 0$, the set of such vectors $(D^2 f(x) v, 0)$ form an $N-1$ dimensional subspace. Call it S.

For any $v \in T_{\nabla f(x)}$, $v \neq 0$, $v' \nabla f(x) = 0$ while $v'[D^2 f(x) v] \neq 0$. This implies that $\nabla f(x) \neq D^2 f(x) v$ for any $v \in T_{\nabla f(x)}$, hence $\nabla f(x) \notin S$. Therefore,

$$A \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array} \right] = \left[\begin{array}{c} \nabla f(x) \\ 0 \end{array} \right]$$

is independent of S.

Finally,

$$A \left[\begin{array}{c} \nabla f(x) \\ 0 \end{array} \right] = \left[\begin{array}{c} D^2 f(x) \nabla f(x) \\ \|\nabla f(x)\|^2 \end{array} \right].$$

which is independent of both S and $(\nabla f(x), 0)$, since $\|\nabla f(x)\| \neq 0$. This establishes that the column space of A is $N+1$ dimensional, hence A has full rank and $|A| \neq 0$. ■

Finally, since a function f is quasi-convex iff $-f$ is quasi-concave, the above results and observations apply also to quasi-convexity, with positive definiteness instead of negative definiteness.