Econ 4111
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Existence of Optima.

1 Introduction.

The mathematics of maximization is the mirror image of the mathematics of minimization: minimizing a function f is the same thing as maximizing the function $-f$. Throughout this course, I exploit this basic symmetry and focus most of my discussion on maximization.

Consider a non-empty set C and a function $f : C \to \mathbb{R}$. C is the set of points that are *feasible* (affordable, physically possible). f is the *objective function* (utility, profits, social welfare). The goal is to find an x^* in C that maximizes f. That is, $f(x^*) \geq f(x)$ for all $x \in C$.

The basic optimization result, Theorem 2, says that a maximum exists provided C is compact and f is continuous. The contrapositive is that if there is no maximum then either C is not compact or f is not continuous.

Suppose, in particular, that $C \subseteq \mathbb{R}^N$. In $\mathbb{R}^N$, recall, a set is compact iff it is closed and bounded. In $\mathbb{R}^N$, therefore, a maximum can fail to exist for one of only three reasons.

1. C may not be bounded. For example, let $C = \mathbb{R}$ and let $f(x) = x$.
2. C may not be closed. For example, let $C = [0, 1)$ and let $f(x) = x$.
3. f may not be continuous. For example, let $C = [0, 1]$ and let

$$f(x) = \begin{cases} x & \text{if } x < 1, \\ 0 & \text{if } x = 1. \end{cases}$$

Theorem 2 gives sufficient, not necessary, conditions for existence of a maximum. For example, suppose $C = \mathbb{R}$ and

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then f is not continuous, C is not compact, but there is a maximum at $x^* = 0$.

2 Existence of a maximum.

Theorem 1. *Let (X, d_x) and (Y, d_y) be metric spaces. For any non-empty set $C \subseteq X$, if $f : C \to Y$ is continuous and C is compact then $f(C)$ is compact.*

Proof. Consider any sequence $\{y_t\}$ in $f(C)$. For each y_t there is an x_t such that $f(x_t) = y_t$ (there may be more than one such x_t if f is not one-to-one). Since C is compact, there is a subsequence of $\{x_t\}$ that converges to a point $x \in C$: $x_{t_k} \to x$. Since f is continuous, $y_{t_k} = f(x_{t_k}) \to f(x)$. Thus, y_{t_k} is a subsequence of $\{y_t\}$ that converges to a point $y = f(x) \in f(C)$. Hence $f(C)$ is (sequentially) compact. ■

I can now state and prove the main result.

Theorem 2. *Let (X, d_x) be a metric space. For any non-empty set $C \subseteq X$, if C is compact and $f : C \to \mathbb{R}$ is continuous then there is an $x^* \in C$ such that $f(x^*) \geq f(x)$ for all $x \in C$.*

Proof. Since C is compact, $f(C)$ is compact and hence it is closed and bounded. Since $f(C) \subseteq \mathbb{R}$ and is bounded above, it has (by the least upper bound property of $\mathbb{R}$) a least upper bound, b. Since $f(C)$ is closed, $b \in f(C)$. Hence there is an x^* such that $f(x^*) = b$ and by construction $f(x^*) \geq f(x)$ for all $x \in C$. ■

Theorem 2 can be strengthened in the following respect. Still assuming C is compact, it is not hard to show that f attains a maximum if it is upper semicontinuous and attains a minimum if it is lower semicontinuous.[1]

3 Maximization in general spaces.

Suppose $X = \mathbb{R}^\infty$ and let $C \subseteq \mathbb{R}^\infty$ consist of points of the form $(0, \dots, 0, 1 - 1/n, 0, \dots)$, where the $1 - 1/n$ term occurs in place n. Consider the problem $\max_{x \in C} f(x)$ where $f(x) = \sup_n x_n$. The finite version of this problem, with N coordinates, has an obvious solution: $x^* = (0, \dots, 0, 1 - 1/N)$. The infinite version has no solution because one can always make the objective function larger by waiting a bit longer. The task is to try to understand what goes wrong in terms of Theorem 2.

Notice that $C \subseteq \ell_\infty$. So I can use either of the two metrics I have introduced for ℓ_∞, namely d_{sup} and d_p. Evidently I must lose either compactness or continuity, or both, under either metric.

Consider first d_{sup}. Under this metric, f is continuous for the trivial reason that any two points in C are roughly distance 1 apart and hence no sequence in C with distinct elements can be Cauchy: f is continuous under d_{sup} because under d_{sup} there are no non-trivial convergent sequences in the domain, and a function is discontinuous only if it jumps at the limit of a convergent sequence.

[1] Let $f : C \to \mathbb{R}$. f is *upper semicontinuous* at x iff for any sequence $x_t \to x$, for any $\varepsilon > 0$, there is a T such that for all $t > T$, $f(x_t) < f(x) + \varepsilon$. The definition of lower semicontinuity is similar, but with $f(x_t) > f(x) - \varepsilon$. A function is continuous iff it is both upper and lower semicontinuous.

On the other hand, under $d_{\sup}$, C is not compact. Consider the sequence $\{x_t\}$. Since, for the reason just given $\{x_t\}$ has no Cauchy subsequences, it has has no convergent subsequence and so C cannot be compact.

Now consider d_p. Under this metric, f is *not* continuous. Again consider the sequence $\{x_t\}$ where $x_t = (0, \dots, 0, 1 - 1/n, 0, \dots)$, with $n = t$. Under d_p, this sequence converges to $0 = \{0, 0, \dots\}$ (remember that convergence under d_p is equivalent to pointwise convergence), and $f(0) = 0$. But $f(x_t) = 1 - 1/t$, which converges to 1.

On the other hand, under d_p, C *is* compact. This follows from Tychonoff's theorem (see the $\mathbb{R}^\infty$ notes), with $C_n = \{0, 1 - 1/n\}$ for each n.

In summary, since this problem has no maximum, either compactness or continuity must fail under any metric that I choose. *Which* of these properties fails, however, can depend on the particular choice of metric.

A related point is that questions of compactness and continuity are in an important sense not intrinsic to the maximization problem. They are auxiliary concepts that we introduce to help us with our analysis. A restatement of Theorem 2 may make this clearer (I don't offer a proof because this *is* just a restatement).

Theorem 3. *Let C be a non-empty subset of a set X and let $f : C \to \mathbb{R}$. If there exists a metric such that C is compact and f is continuous then there is an x^* such that $f(x^*) \geq f(x)$ for all $x \in C$.*

This restatement shifts the emphasis from working with a particular metric that happens to be given to us to instead *finding* a metric that makes C compact and f continuous. To show that a maximum exists it suffices to find *one* such metric, any metric no matter how bizarre. If there is no maximum then there is no such metric. This facet of the maximization problem is obscured in $\mathbb{R}^N$, because in $\mathbb{R}^N$ there is a default metric and we seldom think of looking at alternatives.

Let me conclude with three remarks. First, as I mentioned in the Metric Space notes, there is a generalization of metric spaces called topological spaces that take open sets as primitive. The set of open sets is called a *topology*. It is easy to generalize the definition of compactness and continuity to topological spaces and for such spaces, Theorem 3 becomes the following.

Theorem 4. *Let C be a non-empty subset of a set X and let $f : C \to \mathbb{R}$. If there exists a topology such that C is compact and f is continuous then there is an x^* such that $f(x^*) \geq f(x)$ for all $x \in C$.*

Thus, to prove that $\max_{x \in C} f(x)$ has a solution, find a topology for which C is compact and f is continuous. If no solution exists, then there will be no topology with this property.

Second, there is a tension between compactness and continuity. Say that one topology is stronger than another if the second is a subset of the first (so the first has strictly more open sets than the second). In general, the stronger the topology, the harder it is for C to be compact but the easier it is for $f : C \to \mathbb{R}$ to be continuous. To take an extreme example, in $\mathbb{R}$, two possible topologies are the trivial topology, where only the only open sets are $\emptyset$ and $\mathbb{R}$ itself, and the discrete topology, where every set is open (the trivial topology cannot be generated by a metric). In the trivial topology, every set is compact but the only real-valued functions that are continuous are the constant functions. In the discrete topology, only finite sets are compact but every real-valued function is continuous. In general, there is an art to finding a topology of intermediate strength so that C is compact and f is continuous.

Last, in applications we typically do care about the metric/topology. In particular, we often assume that f is continuous with respect to some topology that has an interpretation that we find sensible. It remains true that we are free to consider other topologies in order to establish existence of a maximum but in practice it often works out that some sensible topology yields existence, assuming that there is a maximum in the first place.