Econ 4111
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Logic and Proofs

1 Sentential Connectives and Tautologies

Formal logical statements are built up out of sentential connectives, such as "and" and "implies," and quantifiers, such as "there exists." To understand the mathematical meaning of the sentential connectives, it is useful to characterize them in terms of truth tables.

I begin with "not." The idea is to assign to some statement α the value T (true) or F (false) and then see how "not" changes that value.

α	not α
T	F
F	T

Next comes "or."

α	β	α or β
T	T	T
T	F	T
F	T	T
F	F	F

Informally, "α or β" is true if and only if either α or β is true. Note that this is the inclusive or: the answer to "coffee or tea?" could be, "both."

Next comes "and."

α	β	α and β
T	T	T
T	F	F
F	T	F
F	F	F

Informally, "α and β" is true if and only if both α and β are true.

Next comes $\rightarrow$, which is a symbol for "implies."

α	β	$\alpha \rightarrow \beta$
T	T	T
T	F	F
F	T	T
F	F	T

Informally, "α implies β" is true *unless* α is true and β is false.[1] In everyday usage, "α implies β" means the same thing as "if α then β" and "α only if β." Note the last. "α if β" is the informal version of $\beta \to \alpha$, *not* of $\alpha \to \beta$.

Finally, consider $\leftrightarrow$, which is the symbol for "if and only if."

α	β	$\alpha \leftrightarrow \beta$
T	T	T
T	F	F
F	T	F
F	F	T

Informally, "α if and only if β" is true if and only if α and β are either both true or both false. I typically abbreviate "if and only if" to iff.

1.1 Tautologies.

A tautology is a statement that is always true, regardless of the truth assignment of its constituent parts.

Using the previous truth tables as building blocks, it is easy to show that

$$(\alpha \to \beta) \leftrightarrow ((\text{not } \beta) \to (\text{not } \alpha))$$

is a tautology. This particular tautology says that $\alpha \to \beta$ is equivalent to its *contrapositive* (not β) $\to$ (not α).

α	β	$\alpha \to \beta$	not β	not α	not $\beta \to$ not α	$(\alpha \to \beta) \leftrightarrow ((\text{not } \beta) \to (\text{not } \alpha))$
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

1.2 Quantifiers

The symbols $\forall$ ("for all" or "for every") and $\exists$ ("there exists") are quantifiers. There are a number of important subtleties involving quantifiers.

The first subtlety is that, in the math that we will be using, a sentence is well formulated only if every variable is bound by a quantifier. In practice this means the following. In English one can say, "Cats are black." This is not well formulated, because no quantifier has been specified for cats. A properly formulated sentence

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[1] Note that $\alpha \to \beta$ is true whenever α is false. Mathematics, in effect, takes the position that $\alpha \to \beta$ is true until proven false. If α is true and β is false then $\alpha \to \beta$ is certainly false. If α is false, however, then we cannot test $\alpha \to \beta$ and so we continue to treat it as true. In practice, any possible discrepancy between everyday language and mathematical usage tends to be moot. One is typically interested in the truth of $\alpha \to \beta$ only when one knows that α is true.

would read "For every cat, the cat is black" (i.e., all cats are black) or "There exists a cat that is black" (i.e., some cats are black). You can also formulate sentences that mean, "At least 50% of cats are black;" I won't torment you with the details about how to do this. Note that if a variable is not bound by a quantifier then the sentence is ambiguous, and may be neither true nor false. We are trying to avoid ambiguity.

A second subtlety is that"$\forall x$" is equivalent to "not $\exists x$ not." For example, the statement that "every cat has fleas" is equivalent to the statement that "there is no cat without fleas." Similarly, "$\exists x$" is equivalent to "not $\forall x$ not." In fact, for formal mathematics, we only need one quantifier, say $\forall$, and we *define* the other quantifier using the above expressions. We use both quantifiers for convenience.

These equivalences come into play when we have to deal with negation, which arises frequently in proofs. For example,"not $\forall x$" is equivalent to "not not $\exists x$ not," which in turn is equivalent to "$\exists x$ not." Thus, the statement that "it is not true that every cat has fleas" is equivalent to the statement that "there is a cat that does not have fleas." Similarly, "not $\exists x$" is equivalent to "$\forall x$ not." For example, the statement that, "there does not exist a cat that weighs as much as the earth" is equivalent to the statement that "for every cat, the cat does not weigh as much as the earth."

More generally, when one takes a negation, then the negation cascades through the quantifiers, flipping $\forall$ and $\exists$, and adding a "not" at the end. For example, negating "$\forall x\ \exists y\ \forall z$" yieilds "$\exists x\ \forall y\ \exists z$ not." Make sure you understand why.

Finally, the order of quantifiers can matter. Consider, for example, the statement,

> For every cat c, there is a flea f such that f is on c.

More colloquially: every cat has fleas. Reversing the quantifiers radically changes the meaning,

> There exists a flea f such that for every cat c, f is on c.

Note that in this statement, it is the *same* flea that is simultaneously on every cat.

A common error is to reverse the order of quantifiers without realizing it. This may be because in standard English it is sometimes possible to reverse quantifiers without changing the listener's or reader's interpretation because the listener or reader effectively corrects your mistake by choosing the interpretation that makes the most sense. For example, if you say "there is a flea on every cat" you will usually be understood to mean the first statement above and not the second. Even in everyday English, however, it is a good idea to get the order of quantifiers correct.

2 Proofs.

Modus ponens ("method of bridging") states: From α and $\alpha \to \beta$, I deduce β. You may view this as a definition of what "deduce" means.

The name *modus ponens* derives from the fact that words like "deduce" belong to the language of ordinary mathematics, not to the special language of logic. Thus, modus ponens provides a bridge between a formalism and the formalism's intended meaning. In particular, *modus ponens* captures the intended meaning of the tautology

$$(\alpha \text{ and } (\alpha \to \beta)) \to \beta.$$

I use the symbol $\Rightarrow$, instead of $\to$, to indicate implication in the ordinary mathematical sense, with *modus ponens* implicit.

In mathematics, a proof is a finite string of applications of *modus ponens*. Starting with a set of statements that I assume to be true, I use *modus ponens* to make a sequence of deductions, ultimately establishing the truth of whatever it is that I was trying to show. Each step is mechanical, making the proof easy to verify. (In practice, people often skip steps, which can make proofs difficult to follow. Also, even when the proof is easy to follow, constructing it in the first place may take great ingenuity.) I refer to such proofs as *deductive proofs*.

Deductive proofs are the *only* proofs used in mathematics. By this I mean that a (seemingly) non-deductive argument is valid if and only if one can transform the non-deductive argument into a deductive proof. In practice, it is common to encounter non-deductive arguments in mathematics. The reason is that non-deductive arguments are often easier to grasp. This is not a mathematical issue (the equivalence with deductive proofs means that there is no mathematical issue) but something to do with the way that human beings think. There are two common forms of non-deductive argument.

- In a proof by contraposition, one proves that $\alpha \Rightarrow \beta$ by showing instead that not $\beta \Rightarrow$ not α. For example, one way to prove that every closed set contains all its limit points is by showing that if a set does not contain all its limit points then it is not closed. Underlying the validity of proofs by contraposition is the contraposition tautology that I demonstrated above.

- In a proof by *reductio ad absurdum* (reducing to absurdity; RAA), one proves that some statement is true by showing that if it were false then there would be an inconsistency, meaning that some other statement would be simultaneously both true and false. Underlying the validity of RAA is the tautology that not $\alpha \to (\beta \leftrightarrow \text{not } \beta)$ iff α. To see this informally, note that $\beta \leftrightarrow \text{not } \beta$ is always false. Therefore, by the truth table definition of $\to$, the only way that not $\alpha \to (\beta \leftrightarrow \text{not } \beta)$ can be true is if not α is false, and hence α is true.

I give an example of these various types of proof in the next section.

One last style of proof that you may have encountered is proof by induction. These are proofs in which one shows that a statement is true for all $n \in \{1, 2, \dots\}$ by showing that it is true for $n = 1$ and then showing that if it is true for n then it is also true for $n + 1$. For example, this is the standard way of showing that

$1 + 2 + \cdots + n = n(n+1)/2$, for all $n \in \{1, 2, \dots\}$. Induction proofs are actually standard deduction proofs; induction is not logically distinct from deduction in the same way that contraposition or RAA are. The "induction" part of the proof is simply a deduction based on a theorem of Set Theory that is in turn deduced from the axioms of Set Theory (axioms are statements assumed to be true).