Economics 4111
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The Implicit Function Theorem

1 Introduction

It is common to encounter functions that are defined only implicitly. Two examples drawn from Economics will help make the point.

1. Consider a standard competitive demand problem. The first order conditions for the consumer's problem, assuming the budget constraint is binding but the non-negativity constraints are not, are

$$\nabla u(x^*) = \lambda^* p^*$$
$$p^* \cdot x^* = m^*$$

   where $x^* \in \mathbb{R}^N_+$ is the demanded bundle, $u : \mathbb{R}^N_+ \to \mathbb{R}$ is the utility function, $\lambda^* \in \mathbb{R}_+$ is the Kuhn-Tucker multiplier on the budget constraint, $p^* \in \mathbb{R}^N_{++}$ is the price vector, and $m^* \in \mathbb{R}_{++}$ is wealth. In specific examples, you may be to solve explicitly for the demand function, giving the demanded bundle as a function of p and m, but even if you cannot, the first order conditions implicitly define the demand function.

2. In an exchange economy, an economy where there is only trade, no production, equilibrium is defined by

$$\bar{f}(p^*, e^*) = 0,$$

   where $p^* \in \mathbb{R}^N_{++}$ is a price vector, $e^* \in \mathbb{R}^{NI}_+$ is the endowment allocation, giving how much of each good each consumer starts with (with I consumers and N goods), and $\bar{f} : \mathbb{R}^N_{++} \times \mathbb{R}^{NI}_+ \to \mathbb{R}^N$ gives the difference between aggregate demand and the aggregate endowment; we are at an equilibrium when this difference is zero. The equality $\bar{f}(p^*, e^*) = 0$ implicitly defines a function giving equilibrium prices as a function of the endowment allocation.

The Implicit Function Theorem is the main mathematical tool for dealing with implicit functions. Given $F : \mathbb{R}^{M+L} \to \mathbb{R}^M$, suppose

$$F(x^*, q^*) = 0$$

with $x^* \in \mathbb{R}^M$ and $q^* \in \mathbb{R}^L$. Interpret q as parameters; we want to understand how x changes as q changes. The Implicit Function theorem says that if F is C^r, $r \geq 1$, in a neighborhood of (x^*, q^*) and if $D_x F(x^*, q^*)$ has full rank (M), then the following hold.

- There is indeed a function ψ that gives x as a function of the parameters q. Specifically, for q near q^*, $\psi(q)$ satisfies the condition $F(\psi(q), q) = 0$.
- The function ψ is itself $\mathcal{C}^r$.

Getting an application to fit the format required by the Implicit Function Theorem often requires some manipulation.

Have established the existence and differentiability of ψ, we can use the Chain Rule to compute $D\psi$ even if we cannot solve for ψ. Explicitly, for any q° near q^*, if $x^\circ = \psi(q^\circ)$, then $F(\psi(q), q) = 0$ implies $D_xF(x^\circ, q^\circ)\psi(q^\circ) + D_qF(x^\circ, q^\circ) = 0$, hence

$$D\psi(q^\circ) = -[D_xF(x^\circ, q^\circ)]^{-1}D_qF(x^\circ, q^\circ).$$

The assumption that $D_xF(x^*, q^*)$ has full rank, and that F is $\mathcal{C}^r$, guarantees that $D_xF(x^\circ, q^\circ)$ is also full rank, and thus invertible.

The formula for $D\psi$ derived above is sometimes conflated with the Implicit Function Theorem. This can lead to confusion since such treatments obscure the importance of establishing the existence and differentiability of ψ.

2 Real-Valued Functions on $\mathbb{R}^2$.

2.1 A Statement of the Theorem.

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a point $x^* \in \mathbb{R}^2$, and let $y^* = f(x^*)$. The *level set* of f through x^* is the set of points x such that $f(x) = y^*$.

The *Implicit Function* theorem states that if f is well behaved at a point x^* then the level set of f through x^* is nice, at least near x^*. More precisely, suppose that f is continuously differentiable and that at least one of the $D_nf(x^*) \neq 0$, so that $Df(x^*) \neq 0$. Then on an open set containing x^*, the level set of f through x^* is the graph of a continuously differentiable function.

In particular, if $D_2f(x^*) \neq 0$, then there are open sets $U \subseteq \mathbb{R}$ and $W \subseteq \mathbb{R}^2$ and a continuously differentiable function $\psi : U \to \mathbb{R}$ such that $D_2f(x) \neq 0$ for any $x \in U$, and

1. $x_1^* \in U$, $x^* \in W$,
2. $\psi(x_1^*) = x_2^*$,
3. For any $x \in W$, $x_1 \in U$,
4. For any $x_1 \in U$, $\psi(x_1)$ is the *unique* x_2 such that, letting $x = (x_1, x_2)$,
   (a) $x \in W$, and
   (b) $f(x) = y^*$

Thus, on the open set W, the level set of f through x^* is the graph of ψ, which is a one-dimensional line or curve. Condition 3 says that U is the projection of W onto the x_1 axis. The simplest form of W is an open rectangle, formed by two open intervals, U and some other interval V, with $W = U \times V$. W can be more irregular than this, however.

The function ψ is said to be defined implicitly by the equation $f(x) = y^*$, hence the name of the theorem. In the next subsection, I give examples of what can go wrong if one or both of the $D_n f(x^*)$ are zero. A general statement of the theorem is in Section 3; the proof is in Section 4.

Suppose $D_2 f(x^*) \neq 0$ and consider any $x \in U$. I can use the Chain Rule to compute, or at least characterize, the slope of ψ, even if I cannot derive ψ explicitly. Define $s : U \to \mathbb{R}$ by $s(x_1) = (x_1, \psi(x_1))$ and $h : U \to \mathbb{R}$ by $h(x_1) = f(s(x_1))$. Then, by construction, $h(x_1) = y^*$ for all $x_1 \in U$ and hence, for all $x_1 \in U$,

$$Dh(x_1) = 0.$$

On the other hand, by the Chain Rule, letting $x = (x_1, \psi(x_1))$,

$$\begin{aligned} Dh(x_1) = Df(x)Ds(x_1) &= \begin{bmatrix} D_1 f(x) & D_2 f(x) \end{bmatrix} \begin{bmatrix} 1 \\ D\psi(x_1) \end{bmatrix} \\ &= D_1 f(x) + D_2 f(x) D\psi(x_1). \end{aligned}$$

Combining,

$$0 = D_1 f(x) + D_2 f(x) D\psi(x_1). \tag{1}$$

Since $D_2 f(x) \neq 0$, by property 5,

$$D\psi(x_1) = -\frac{D_1 f(x)}{D_2 f(x)}. \tag{2}$$

Equation 2 is sometimes referred to *as* the Implicit Function theorem, but, as already discussed in the introduction, this is wrong. Equation 2 is just a routine application of the Chain Rule. The Implicit Function theorem is used to guarantee that ψ exists and is differentiable.

2.2 Examples.

The following examples provide illustrations both of how the Implicit Function theorem works and what can go wrong. It should be reasonably intuitive that if f is not differentiable then ψ may not be either. Therefore, in all of the following examples, f is differentiable, in fact $\mathcal{C}^\infty$.

Example 1. Define $f^A : \mathbb{R}^2 \to \mathbb{R}$ by

$$f^A(x) = x_1 x_2.$$

Let $x^* = (2, 1/2)$, hence $y^* = f(x^*) = 1$. Since $D_2 f^A(x^*) = x_1^* = 2 \neq 0$, the assumptions of the Implicit Function theorem are satisfied. In fact, it is easy to find U, W, and ψ explicitly. Let $U = \mathbb{R}_{++}$, let $W = \mathbb{R}_{++} \times \mathbb{R}$, and let $\psi : \mathbb{R}_{++} \to \mathbb{R}$ be given by

$$\psi(x_1) = \frac{1}{x_1}.$$

The graph of this, the level set of f^A, is a rectangular hyperbola.

By direct calculation

$$D\psi(A)(x_1) = -\frac{1}{x_1^2}.$$

Alternatively, by Equation 2, at the point $x = (x_1, \psi(x_1))$,

$$D\psi(x_1) = -\frac{D_1 f^A(x)}{D_2 f^A(x)} = -\frac{x_1}{x_2} = -\frac{\psi(x_1)}{x_1} = -\frac{1}{x_1^2}.$$

So Equation 2 works.

Interpreting f^A as a utility function, the level set of f is an indifference curve and $D\psi$ is the slope of this indifference curve. The absolute value of this slope is usually called the Marginal Rate of Substitution (MRS) at x^*.

Note that I cannot take U, the domain of ψ, to be $\mathbb{R}$, even though f itself is defined on all of $\mathbb{R}$. If $x_1 = 0$ then for any x_2, $f(x_1, x_2) = 0 \neq 1$. In defining ψ, one often has to restrict its domain to some proper subset of the original domain of f. □

Example 2. Define $f^B : \mathbb{R}^2 \to \mathbb{R}$ by

$$f^B(x) = x_1^2 + x_2^2.$$

Let $x^* = (0, 1)$, hence $y^* = f^B(x^*) = 1$. The level set defined by $f^B(x) = 1$ is the circle of radius one centered at the origin, with x^* on the upper half of this circle.

Since $D_2 f^B(x^*) = 2x_2^* = 2 \neq 0$, the assumptions of the Implicit Function theorem are satisfied. In fact, it is easy to find U, W, and ψ explicitly. Take $U = (-1, 1)$ and $W = (-1, 1) \times \mathbb{R}_{++}$. Then

$$\psi(x_1) = \sqrt{1 - x_1^2}.$$

There are some subtleties here.

1. As in the previous example, the domain of ψ has to be restricted to a proper subset of the domain of f^B.

2. The domain of ψ *can* be extended to $\bar{U} = [-1, 1]$, *but* if I do so then ψ is not differentiable at the endpoints.

3. At $\hat{x} = (1, 0)$, although $D_2 f^B(\hat{x}) = 0$, $D_1 f^B(\hat{x}) = 2 \neq 0$. So I can represent the level set of f^B through $\hat{x}$ as the graph of a differentiable function from x_2 to x_1 (rather than from x_1 to x_2). In particular, I can use $\hat{U} = (-1, 1)$, $\hat{W} = \mathbb{R}_{++} \times (-1, 1)$, and the function $\hat{\psi} : U \to \mathbb{R}_{++}$ defined by

$$\hat{\psi}(x_2) = \sqrt{1 - x_1^2}.$$

4. At $x^* = (0, 1)$, one cannot replace W with the larger set $\tilde{W} = (-1, 1) \times \mathbb{R}$. At $x_1^* = 0$, for example, there are two values of x_2 that satisfy $f^B(x_1^*, x_2) = 1$, namely $\psi(x_1^*) = x_2^* = 1$ and also $x_2 = -1$. The graph of ψ equals the level set of f^B restricted to W but not the level set of f^B restricted only to the larger set $\tilde{W}$.

   More generally, the level set of f^B, which is a circle, cannot be represented as the graph of any one function, even if I drop the differentiability requirement. To represent the level set of f^B while maintaining the differentiability requirement, I need a minimum of four implicit functions and corresponding W. For example, I can use ψ, $\hat{\psi}$, already given, and then two other functions to pick up points such as $(0, -1)$ and $(-1, 0)$.

□

Example 3. Again consider f^A but now let $x^* = (0, 0)$. Now we have a problem because

$$Df(x^*) = [\ 0 \quad 0\ ],$$

which violates the condition of the Implicit Function theorem. The level set of f^B through x^* resembles a "+" sign. There is no way to represent this level set as the graph of a function, differentiable or otherwise. □

Example 4. Define $f^C : \mathbb{R}^2 \to \mathbb{R}$ by

$$f^C(x) = (x_1 - x_2)^2.$$

Then at $x^* = (0, 0)$,

$$Df^C(x^*) = \left[\ 2(x_1^* - x_2^*) \quad -2(x_1^* - x_2^*)\ \right] = [\ 0 \quad 0\ ].$$

Here, however, a $\mathcal{C}^\infty$ ψ *does* exist, with $U = \mathbb{R}$ and $W = \mathbb{R}^2$:

$$\psi(x_1) = x_2.$$

So the condition $Df(x^*) \neq 0$ is not *necessary* for differentiability of the implicit function ψ. In contrast, for the Inverse Function theorem, invertibility of the derivative of the function *was* necessary for the differentiability of the inverse function. □

3 The General Case.

Consider a function $f : \mathcal{O} \to \mathbb{R}^M$ where $\mathcal{O}$ is an open subset of $\mathbb{R}^{L+M}$. It is convenient to denote a point $x \in \mathbb{R}^{L+M}$ as (x_λ, x_μ), where x_λ denotes a point in $\mathbb{R}^L$ and x_μ denotes a point in $\mathbb{R}^M$. At a point x^*, let $D_\lambda f(x^*)$ denote the first L columns of $Df(x^*)$ (the x_λ columns) and let $D_\mu f(x^*)$ denote the remaining M columns (the x_μ columns).

Theorem 1 (Implicit Function theorem)**.** *Let $\mathcal{O}$ be a nonempty open subset of $\mathbb{R}^{L+M}$. Let $f : \mathcal{O} \to \mathbb{R}^M$ be $\mathcal{C}^r$, where r is a positive integer. Fix $x^* \in \mathcal{O}$ and let $f(x^*) = y^*$.*

If $Df(x^*)$ has full rank, namely M, then there is an open set W in $\mathbb{R}^{L+M}$ such that the restriction of the level set $f^{-1}(y^*)$ to W is the graph of a $\mathcal{C}^r$ function.

In particular, suppose, for concreteness and simplicity of notation, that the last M columns of $Df(x^*)$ (the x_μ columns) are linearly independent. Then there are open sets $U \subseteq \mathbb{R}^L$ and $W \subseteq \mathbb{R}^{L+M}$, and a $\mathcal{C}^r$ function $\psi : U \to \mathbb{R}^M$ such that, $D_\mu f(x)$ has full rank for all $x \in U$, and

1. *$x^*_\lambda \in U$, $x^* \in W$,*
2. *$\psi(x^*_\lambda) = x^*_\mu$,*
3. *For any $x \in W$, $x_\lambda \in U$,*
4. *For any $x_\lambda \in U$, $\psi(x_\lambda)$ is the* unique *x_μ such that, letting $x = (x_\lambda, x_\mu)$,*
   - *(a) $x \in W$,*
   - *(b) $f(x) = y^*$.*

Proof. See Section 4. ■

The Implicit Function theorem thus states that if f is continuously differentiable and $Df(x^*)$ has full rank then the level set of f through x^* is, near x^*, an L-dimensional surface in $\mathbb{R}^{L+M}$. Example 3 in Section 2.2 shows what can go wrong.

The proof of the Implicit Function theorem is an application of the Inverse Function Theorem; the Implicit Function can be viewed as a corollary. Accordingly, the subtlety involving continuous differentiability that I discussed for the Inverse Function theorem applies here as well.

I can use the Chain Rule to compute $D\psi$ even if I can't compute ψ directly. Suppose that $D_\mu f(x^*)$ has full rank. Define $s : U \to \mathbb{R}^{L+M}$ by $s(q) = (q, \psi(q))$. Define $h : U \to \mathbb{R}^M$, by $h(x_\lambda) = f(s(x_\lambda))$. Then $h(x_\lambda) = y^*$ for all $x_\lambda \in U$, hence

$$Dh(x_\lambda) = 0.$$

On the other hand, by the Chain Rule, let $x = (x_\lambda, \psi(x_\lambda))$,

$$\begin{aligned} Dh(x_\lambda) &= Df(x)Ds(x_\lambda) \\ &= \begin{bmatrix} D_\lambda f(x) & D_\mu f(x) \end{bmatrix} \begin{bmatrix} I \\ D\psi(x_\lambda) \end{bmatrix} \\ &= D_\lambda f(x) + D_\mu f(x) D\psi(x_\lambda) \end{aligned}$$

where I is the $L \times L$ identity matrix. Putting all this together

$$0 = D_\lambda f(x) + D_\mu f(x) D\psi(x_\lambda).$$

Since $D_\mu f(x^*)$ has full rank, since f is continuously differentiable, and since the determinant is continuous, $D_\mu f(x)$ has full rank for all x in a ball around x^*. For any such x,

$$D\psi(x_\lambda) = -\left[D\mu f(x)\right]^{-1} D_\lambda f(x). \tag{3}$$

This is a generalization of Equation 2.

As with Equation 2, you may sometimes see Equation 3 referred to as being the Implicit Function theorem. Again, this is incorrect: Equation 3 is just a routine application of the Chain Rule. The Implicit Function theorem is what allows us to conclude that the function ψ exists and is differentiable.

The statement of Theorem 1 can be strengthened in a number of directions. In particular, I can extend the proof to show that there is an implicit function ψ_y for each y in an open set containing y^*. Moreover, I can take each of these ψ_y to have the same U as its domain.

4 Proof of the Implicit Function Theorem.

Define $F : \mathcal{O} \to \mathbb{R}^N$ by $F(x) = (x_\lambda, f(x))$. Then F is $\mathcal{C}^r$ and

$$DF(x) = \begin{bmatrix} I & \mathbf{0} \\ D_\lambda f(x) & D_\mu f(x) \end{bmatrix},$$

where I is the $L \times L$ identity and $\mathbf{0}$ is the $L \times M$ matrix of zeroes.

$F(x^*) = (x^*_\lambda, y^*)$. Moreover, $DF(x^*)$ has full rank, since $|DF(x^*)| = |D_\mu f(x^*)|$ and $|D_\mu f(x^*)| \neq 0$ by assumption. Therefore, by the Inverse Function theorem, there is an open set $\tilde{\mathcal{O}} \subseteq \mathcal{O}$, with $x^* \in \tilde{\mathcal{O}}$, and an open set $V \subseteq \mathbb{R}^{L+M}$, with $(x^*_\lambda, y^*) \in V$, such that $DF(x)$ has full rank for every x in $\tilde{\mathcal{O}}$, F maps $\tilde{\mathcal{O}}$ 1-1 onto V and the inverse $\Psi : V \to \tilde{\mathcal{O}}$ is $\mathcal{C}^r$.

Since V is open, there exists an open set $U \subseteq \mathbb{R}^L$ such that $x^*_\lambda \in U$ and, for every $x_\lambda \in U$, $(x_\lambda, y^*) \in V$.[1] Since $D_\mu f(x^*)$ has full rank, f is continuously differentiable,

[1]As discussed in the notes on $\mathbb{R}^N$, any open ball is contained in an open cube and vice versa. In the present case, since V is open and contains (x^*_λ, y^*), there is an $\varepsilon > 0$ such $N_\varepsilon(x^*_\lambda, y^*) \subseteq V$. Choose any $r > 0$ such that $r\sqrt{L+M} \leq \varepsilon$. Then the $L+M$-dimensional cube with sides of length $2r$ and centered at (x^*_λ, y^*) is contained in $N_\varepsilon(x^*_\lambda, y^*)$, which is contained in V. Take U to be the L-dimensional cube with sides of length $2r$ and centered at x^*_λ.

and the determinate function is continuous, one can take U such that $D_\mu f(x)$ has full rank for any $x \in U$. Let $W = \left(U \times \mathbb{R}^M\right) \cap \tilde{\mathcal{O}}$. This is open, since it is the intersection of two open sets.

For any $x_\lambda \in U$, the last M coordinates of $\Psi(x_\lambda, y^*)$ identifies the unique point $x_\mu \in R^M$ such that, setting $x = (x_\lambda, x_\mu)$, $x \in \tilde{\mathcal{O}}$, and $f(x) = y^*$. Moreover, by construction, $x \in W$. Therefore, define $\psi : U \to \mathbb{R}^M$ by setting $\psi(x_\lambda)$ equal to the last M coordinates of $\Psi(x_\lambda, y^*)$. Since Ψ is $\mathcal{C}^r$ on A, ψ is $\mathcal{C}^r$ on U. ■