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Definite Matrices

1 Basic Definition

An $N \times N$ *symmetric* matrix A is *positive definite* iff for any $v \neq 0$, $v'Av > 0$. For example, if

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

then the statement is that for any $v = (v_1, v_2) \neq 0$,

$$\begin{aligned} v'Av &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} av_1 + bv_2 \\ bv_1 + cv_2 \end{bmatrix} \\ &= av_1^2 + 2bv_1v_2 + cv_2^2 \\ &> 0 \end{aligned}$$

(The expression $av_1^2 + 2bv_1v_2 + cv_2^2$ is called a *quadratic form*.) An $N \times N$ symmetric matrix A is *negative definite* iff $-A$ is positive definite. The definition of a positive *semi*definite matrix relaxes $>$ to $\geq$, and similarly for negative semi-definiteness.

If $N = 1$ then A is just a number and a number is positive definite iff it is positive. For $N > 1$ the condition of being positive definite is somewhat subtle. For example, the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

is not positive definite. If $v = (1, -1)$ then $v'Av = -4$. Loosely, a matrix is positive definite iff (a) it has a diagonal that is positive and (b) off diagonal terms are not too large in absolute value relative to the terms on the diagonal. I won't formalize this assertion but it should be plausible given the example. The canonical positive definite matrix is the identity matrix, where all the off diagonal terms are zero.

A useful fact is the following. If S is any $M \times N$ matrix then $A = S'S$ is positive semi-definite. To see this, note that $S'S$ is symmetric $N \times N$. To see that it is positive semi-definite, note that for any $v \in \mathbb{R}^N$,

$$v'Av = v'[S'S]v = (v'S')(Sv) = (Sv)'(Sv) \geq 0.$$

Moreover, by this argument, $v'Av = 0$ iff $Sv = 0$. This cannot happen if S has full column rank. Thus $S'S$ is positive definite, not merely positive semi-definite, if the rank of S is N, which implies $N \leq M$.

The converse of all this is also true, although I will not establish it. If A is positive definite then there is a full rank $N \times N$ matrix S such that $A = S'S$. If A is negative semi-definite and has rank $M \leq N$ then there is an $M \times N$ matrix of rank M such that $A = S'S$.

2 Inverses of definite matrices.

Theorem 1. *If A is positive definite then A is invertible and A^{-1} is positive definite.*

Proof. If A is positive definite then $v'Av > 0$ for all $v \neq 0$, hence $Av \neq 0$ for all $v \neq 0$, hence A has full rank, hence A is invertible.

For any invertible matrix A, $(A^{-1})' = (A')^{-1}$. To see this, note that $[A'(A^{-1})']' = A^{-1}A = I$. Hence $A'(A^{-1})' = I' = I$. Similarly, $(A^{-1})'A' = I$.

If A is symmetric and invertible then $(A^{-1})' = (A')^{-1} = A^{-1}$, hence A^{-1} is symmetric. It remains to show that A^{-1} is positive definite. Consider $v'A^{-1}v$ for any $v \neq 0$. Then $v'A^{-1}v = v'A^{-1}AA^{-1}v = (A^{-1}v)'A(A^{-1}v) > 0$, where the second equality comes from the fact that A^{-1} is symmetric and the last inequality follows from the fact that A is positive definite. ■

Likewise, if A is negative definite then A^{-1} exists and is negative definite.

3 Checking definiteness.

There is a mechanical check for definiteness of a matrix. It is messy but easy to implement on a computer.

Given an $N \times N$ matrix A, a *leading principal submatrix* of A is a submatrix formed by deleting all but the first n rows and columns. Thus, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then the leading principal submatrices are

$$\begin{bmatrix} a_{11} \end{bmatrix},$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and A itself. A *leading principal minor* is the determinant of a leading principal submatrix.

Theorem 2. *A matrix is positive definite iff all of the leading principal minors are positive.*

Proof. Omitted. ■

Since A is negative definite iff $-A$ is positive definite, and since multiplying an $n \times n$ matrix by -1 multiplies the determinant by $(-1)^n$, Theorem 2 implies that a matrix is negative definite iff the principal minors alternate in sign, with the sign negative iff the number of columns in the submatrix of A is odd.

Note that Theorem 2 implicitly assumes that the matrix is symmetric. If the matrix is not symmetric then the leading principal minor condition does not guarantee that $v'Av > 0$ for all $v \neq 0$. For example, consider

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix},$$

This matrix passes the principal minor test, but for $v = (1, -1)$, $v'Av = -8$.