Econ 4111
Professor: John Nachbar
December 7, 2011

Convex Sets, Separation, and Support in $\mathbb{R}^N$

1 Convex Sets

Much of the discussion here generalizes to infinite dimensional vector spaces but for concreteness I focus on subsets of $\mathbb{R}^N$.

Definition 1. *A set $S \subseteq \mathbb{R}^N$ is* convex *iff for any $a, b \in S$ and any $\theta \in [0, 1]$ the point $\theta a + (1 - \theta)b$ is also in S.*

Geometrically, S is convex iff S contains the line segment joining any two points in S. Thus, a neighborhood is convex (see below) but the letter C is not. See figure 1.

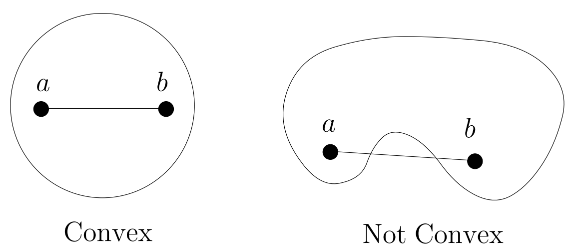

Figure 1: Convex and non-convex sets

Given two sets (not necessarily convex) A and B, $A + B$ denotes the set formed by adding all possible combinations of points in A and B

Definition 2. *Let A and B be sets in $\mathbb{R}^N$. Let $r \in \mathbb{R}$.*

- $A + B = \{x \in \mathbb{R}^N : \exists\, a \in A,\ b \in B$ *such that* $a + b = x\}$.
- $rA = \{x \in \mathbb{R}^N : \exists\, a \in A$ *such that* $x = ra\}$.

Note: In these notes, $A - B$ means $A + (-B)$ *not* $A \setminus B$.

Theorem 1. *Let $A, B \subseteq \mathbb{R}^N$ be convex.*

1. *$A + B$ is convex.*

2. *$r\,A$ is convex, for any $r \in \mathbb{R}$.*

3. *$A \cap B$ is convex.*

4. *$\overline{A}$ is convex.*

Proof. I'll prove that $\overline{A}$ is convex, which is the least straightforward. I leave the others as exercises. Consider any $a, \hat{a} \in \overline{A}$ and any $\theta \in [0, 1]$. Let $x = \theta a + (1 - \theta)\hat{a}$. Take sequences $\{a_t\}$ and $\{\hat{a}_t\}$ in A such that $a_t \to a$ and $\hat{a}_t \to \hat{a}$. (If $a \in A$ then one can take $a_t = a$ and similarly for $\hat{a}$.) Let $x_t = \theta a_t + (1 - \theta)\hat{a}_t$. Since A is convex, $x_t \in A$. By continuity $x_t \to x$. Since $\overline{A}$ is closed, $x \in \overline{A}$. ■

Remark 1. If $r > 1$ then A looks like an inflated version of A. If $0 < r < 1$ then $r\,A$ looks like a deflated version of A. If A is negative then r will both inflate/deflate A and reflect it through the origin. □

Theorem 2. *For any $x \in \mathbb{R}^N$ and any $\varepsilon > 0$, $N_\varepsilon(x)$ is convex.*

Proof. Since $N_\varepsilon(x) = N_\varepsilon(0) + \{x\}$, it suffices to show that $N_\varepsilon(0)$ is convex. Consider any $a, b \in N_\varepsilon(0)$ and any $\theta \in [0, 1]$. I must show that $\|\theta a + (1 - \theta)b\| < \varepsilon$. By the basic properties of the Euclidean norm,

$$\begin{aligned}\|\theta a + (1 - \theta)b\| &\leq \theta\|a\| + (1 - \theta)\|b\| \\ &< \theta\varepsilon + (1 - \theta)\varepsilon \\ &= \varepsilon,\end{aligned}$$

as was to be shown. ■

In the special case of $\mathbb{R}$, S is convex iff it is an interval (possibly infinite, not necessarily containing one or both endpoints).

2 A Basic Separation Result.

The major separation and support theorems are consequences of the more basic fact, recorded in Theorem 3 below, that if a closed convex set does not contain zero then it is possible to find a plane through the origin that lies entirely to one side of the convex set. An $N - 1$ dimensional plane through the origin can always be described in the form $T_v = \{x \in \mathbb{R}^N : v \cdot x = 0\}$ for some $v \neq 0$. Planes are often called *hyperplanes*, especially if the dimension N is either unspecified or greater than 3. The left-hand pane of Figure 2 provides an example in which a convex set E is separated from the origin in this way. The right-hand pane shows that if E is not convex then there may be no plane through the origin that has E entirely to one side.

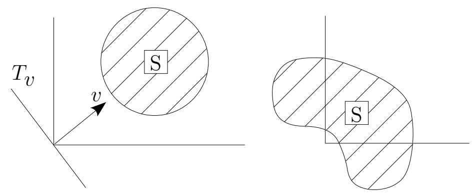

Figure 2: Separation from the origin.

Theorem 3 (The Basic Separation Theorem.). *Let $S \subseteq \mathbb{R}^N$ be a nonempty closed, convex set such that $0 \notin S$. Then there is a $v \neq 0$ such that $v \cdot x \geq v \cdot v$ for all $x \in S$. In particular, $v \cdot x > 0$ for all $x \in S$.*

I first sketch the basic idea of the proof. Take v to be the element of E that is closest to 0, which is to say that v is the element of E with smallest norm. See Figure 3. The circle shows those points that have norm equal to $\|v\|$. Now, suppose

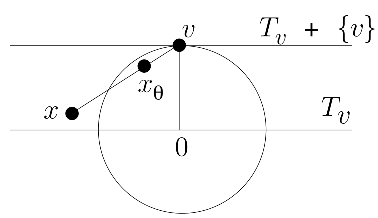

Figure 3: The intuition for Theorem 3

there is some $x \in E$ with $v \cdot x \leq v \cdot v$. In the picture, these are the points that lie below the line $T_v + \{v\}$. As the picture shows, I can find an $x_\theta = \theta x + (1 - \theta)v$ between x and v that has smaller norm (lies within the circle). And since E is convex, $x_\theta \in E$. This contradicts the assumption that v is an element of E with smallest norm. A formal proof is as follows.

Proof. I first claim that there exists a solution $v \neq 0$ to the problem

$$\min_{x \in S} \|x\|.$$

To see this, note that the function $\|\cdot\|$ is continuous. A solution would therefore exist if S were compact. S is not necessarily compact, but this is not really a problem. Since S is not empty, there is some $z \in S$, $z \neq 0$ (since $0 \notin S$). Take $\hat{S} = S \cap \overline{N_{\|z\|}(0)}$.

Then $\hat{S}$ is non-empty and compact and so the problem $\min_{x \in \hat{S}} \|x\|$ has a solution. Call this solution v. By construction, the norm of v is smaller than the norm of any x in $S \setminus \hat{S}$, so v solves the original minimization problem, $\min_{x \in S} \|x\|$, as well. Since $0 \notin S$, $v \neq 0$.

I claim that $v \cdot x \geq v \cdot v$ for any $x \in S$. I argue by contraposition. Suppose that $x \in S$ and $v \cdot x < v \cdot v$, hence $v \cdot (x - v) < 0$. One can verify, using the definition of $\| \cdot \|$, that the derivative of $\| \cdot \|$ at v in the direction $x - v$ is

$$\lim_{\theta \to 0} \frac{\|v + \theta(x - v)\| - \|v\|}{\theta} = \frac{v \cdot (x - v)}{\|v\|} < 0.$$

This implies that for θ close to 0, setting $x_\theta = \theta x + (1 - \theta)v = v + \theta(x - v)$,

$$\frac{\|x_\theta\| - \|v\|}{\theta} < 0,$$

hence $\|x_\theta\| < \|v\|$. Since v mimizes $\|x\|$ on S, this implies that $x_\theta \notin S$. Since $x, v \in S$, this shows that S is not convex. By contraposition, if S is convex, then $v \cdot x \geq v \cdot v$ for all $x \in S$. ■

3 Support.

A set S is supported at x^* iff S lies to one side of an $N - 1$ dimensional hyperplane passing through x^*. Any such hyperplane can be represented in the form $\{x \in \mathbb{R}^N : v \cdot x = v \cdot x^*\} = T_v + \{x^*\}$ for some $v \neq 0$. See Figure 4. As drawn, x^* is a boundary

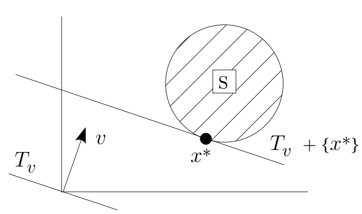

Figure 4: A supporting plane.

point of E.

Definition 3. *A nonempty set $S \subseteq \mathbb{R}^N$ is* supported *at x^* iff $\exists v \neq 0$ such that $v \cdot x \geq v \cdot x^*$ for all $x \in S$.*

Recall that x is an interior point of S if there exists an $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq S$. The *interior* of S, denoted int S, is the set of interior points of S. If $x \notin \text{int } S$ and S is closed then either $x \in S^c$ or x is a boundary point of S, meaning $x \in S \cap \overline{S^c}$.

Theorem 4 (Supporting Hyperplane Theorem)**.** *Let $S \subseteq \mathbb{R}^N$ be a nonempty closed and convex set. If $x^* \notin \text{int } S$, then S is supported at x^*.*

The proof of both Theorem 4 and Theorem 6 depend on the following fact about sums of closed sets.

Theorem 5. *Let $A, B \subseteq \mathbb{R}^N$ be closed. If at least one set is compact then $A + B$ is closed.*

Proof of Theorem 5. Suppose $\{c_t\}$ is a sequence in C that converges, say to c. I must check that $c \in C$. Since $c_t \in C$ there is an $a_t \in A$ and a $b_t \in B$ such that $a_t + b_t = c_t$. By assumption, one of A or B is compact. Without loss of generality, suppose that A is compact. Then $\{a_t\}$ has a subsequence that converges to a point of A. Call this subsequential limit a. Along this subsequence, continuity implies that $b_t = a_t - c_t$ converges to $a - c$. Define $b = a - c$. Since B is closed, $b \in B$. Therefore, $c = a + b \in A + B = C$, as was to be shown. ■

Example 1. If A and B are closed but neither is compact then $C = A + B$ may not be closed. For example, let $A = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 \geq 1/x_1\}$ and let $B = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Then A and B are closed but $C = A + B$ is not closed. In fact, C is the half space $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$; every point on the x_1 axis is a limit point of C but no point on the x_1 axis is an element of C. □

Proof of Theorem 4. If $x^* \notin S$ then, since S is closed, the proof follows from Theorem 3; in fact, in this case, S, is *strictly* supported at x^*. The hard case is when x^* is a boundary point of S.

Since $x^* \notin \text{int } S$, for any $\varepsilon > 0$ there is a $x_\varepsilon \in N_\varepsilon(x^*) \cap S^c$. This implies that there is a sequence $\{x_t\}$ in S^c converging to x^*. For each x_t, consider the set $C_t = S - \{x_t\}$ (where $\{x_t\}$ is the set with one element, the point x_t). Since S is closed and convex and $\{x_t\}$ is (trivially) compact and convex, C_t is closed and convex. Moreover, since $x_t \notin S$, $0 \notin C_t$. Apply Theorem 3 to get $v_t \in C_t$ such that $v_t \cdot c > 0$ for all $c \in C_t$. Since $c \in C_t$ iff there is an $x \in S$ such that $c = x - x_t$, this implies that for all $x \in S$, $v \cdot (x - x_t) > 0$, hence $v_t \cdot x > v_t \cdot x_t$.

Unfortunately, there is no guarantee that v_t converges to anything. This is easily fixed, however. Multiply both sides of $v_t \cdot x > v_t \cdot x_t$ by $1/\|v_t\| > 0$ to get

$$\hat{v}_t \cdot x > \hat{v}_t \cdot x_t,$$

where $\hat{v}_t = v_t/\|v_t\|$. Then $\hat{v}_t$ belongs to a compact set, namely the unit circle $\{v \in \mathbb{R}^N : \|x\| = 1\}$. Because the unit circle is compact, $\{\hat{v}_t\}$ has a convergent subsequence, converging to, say, v on the unit circle.

By continuity of inner product it follows that, for all $x \in S$,

$$v \cdot x \geq v \cdot x^*,$$

which is what I needed to show. ■

Theorem 4 can be generalized to handle S that is not closed but this case is more difficult.

4 Separation.

Definition 4. *Consider two nonempty sets $A, B \subseteq \mathbb{R}^N$. A and B can be* strictly separated by a hyperplane *iff there exists a $v \in \mathbb{R}^N$, $v \neq 0$, and an $r \in \mathbb{R}$ such that $v \cdot a > r > v \cdot b$ for every $a \in A, b \in B$.*

In Figure 5, the line drawn is a hyperplane strictly separating A and B.

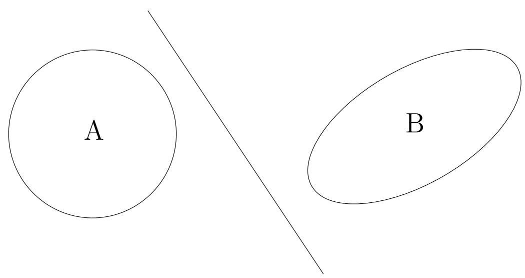

Figure 5: A separating hyperplane.

Theorem 6 (Separating Hyperplane Theorem)**.** *Suppose that $A, B \subseteq \mathbb{R}^N$ are nonempty closed, and convex, with at least one set compact. If $A \cap B = \emptyset$ then A and B can be strictly separated by a hyperplane.*

Proof. Consider $C = A - B$. Since A and B are convex, Theorem 1 implies that C is convex. And since A and B are closed and at least one is compact, Theorem 5 implies that C is closed.

Since $A \cap B = \emptyset$, $0 \notin C$. Applying Theorem 3, there is a $v \neq 0$ such that $v \cdot x \geq v \cdot v > 0$ for all $x \in C$. This implies that $v \cdot a \geq v \cdot b + v \cdot v$ for all $a \in A$ and all $b \in B$. Define

$$\overline{r} = \inf\{r \in \mathbb{R} : \exists a \in A \text{ such that } v \cdot a = r\},$$
$$\underline{r} = \sup\{r \in \mathbb{R} : \exists b \in B \text{ such that } v \cdot b = r\}.$$

Since $v \cdot a \geq v \cdot b + v \cdot v$ for every $a \in A$, $b \in B$, it follows that $\overline{r} \geq v \cdot b + v \cdot v$ for any $b \in B$, hence $\overline{r} \geq \underline{r} + v \cdot v > \underline{r}$. Choose any $r \in (\underline{r}, \overline{r})$ and then $v \cdot a > r > v \cdot b$ for any $a \in A$, $b \in B$, as was to be shown. ■

Theorem 6 can be extended to A and B that are convex but not necessarily closed, let alone compact. The extension yields only weak separation, rather than strict separation.

Example 2. Strict separation can fail if neither set is compact, even if both are closed. As in Example 1, suppose $A = \{x \in \mathbb{R}^2 : x_1 > 0 \text{ and } x_2 \geq 1/x_2\}$ and $B = \{x \in R^2 : x_2 = 0\}$. To separate A and B, v must be collinear with $(0, 1)$. Without loss of generality, suppose $v = (0, 1)$. Then $v \cdot b = 0$ for all $b \in B$. On the other hand, $\inf\{q \in \mathbb{R} : \exists a \in A \text{ such that } v \cdot a = q\} = 0$. This implies that there cannot be strict separation. □

Example 3. Strict separation can also fail if one of the sets is not closed, even if that set is bounded and the other set is compact. For example, let $A = (1, 2)$ and let $B = [0, 1]$. Clearly we cannot have strict separation. □