Econ 4111
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Concave and Convex Functions

1 Basic Definitions.

Definition 1. *Let $C \subseteq \mathbb{R}^N$ be convex and let $f : C \to \mathbb{R}$.*

1. (a) *f is* concave *iff for any $a, b \in C$ and any $\theta \in [0, 1]$,*

$$f(\theta a + (1-\theta)b) \geq \theta f(a) + (1-\theta)f(b);$$

   (b) *f is* strictly concave *iff for any $a, b \in C$ and any $\theta \in (0, 1)$, the above inequality is strict.*

2. (a) *f is* convex *iff for any $a, b \in C$ and any $\theta \in [0, 1]$,*

$$f(\theta a + (1-\theta)b) \leq \theta f(a) + (1-\theta)f(b);$$

   (b) *f is* strictly convex *iff for any $a, b \in C$ and any $\theta \in (0, 1)$, the above inequality is strict.*

The following equivalence is immediate from the definitions.

Theorem 1. *Let $C \subseteq \mathbb{R}^N$ be convex and let $f : C \to \mathbb{R}$. f is convex iff $-f$ is concave. f is strictly convex iff $-f$ is strictly concave.*

It is also possible to characterize concavity or convexity of functions in terms of the convexity of particular sets. Given the graph of a function, the *hypograph* of f, written hyp$f$, is the set of points that lies on or below the graph of f, while the *epigraph* of f, written epi$f$, is the set of points that lies on or above the graph of f.[1] Formally,

$$\text{epi}f = \{(x, y) \in \mathbb{R}^{N+1} : y \geq f(x)\},$$
$$\text{hyp}f = \{(x, y) \in \mathbb{R}^{N+1} : y \leq f(x)\}.$$

Theorem 2. *Let $C \subseteq \mathbb{R}^N$ be convex and let $f : C \to \mathbb{R}$.*

1. *f is concave iff* hyp$f$ *is convex.*
2. *f is convex iff* epi$f$ *is convex.*

[1]"Hypo" means "under" and "epi" means "over." A hypodermic needle goes under your skin, the top layer of which is your epidermis.

Proof. Suppose that f is concave. I will show that hyp$f$ is convex. Take any $z_1, z_2 \in \text{hyp}f$ and any $\theta \in [0,1]$. Then there is an $a, b \in C$ and $y_1, y_2 \in \mathbb{R}$, such that $z_1 = (a, y_1)$, $z_2 = (b, y_2)$, with $f(a) \geq y_1$, $f(b) \geq y_2$. By concavity of f, $f(\theta a + (1-\theta)b) \geq \theta f(a) + (1-\theta)f(b)$. Hence $f(\theta a + (1-\theta)b) \geq \theta y_1 + (1-\theta)y_2$. The latter says that the point $\theta z_1 + (1-\theta)z_2 = (\theta a + (1-\theta)b, \theta y_1 + (1-\theta)y_2) \in \text{hyp}f$, as was to be shown. The other directions are similar. ■

2 Concavity, Convexity, and Continuity.

Theorem 3. *Let $C \subseteq \mathbb{R}^N$ be open and convex and let $f : C \to \mathbb{R}$ be either convex or concave. Then f is continuous.*

Proof. See Rockafellar (1970). ■

For functions defined on non-open sets, continuity can fail at the boundary. In particular, if the domain is a closed interval in $\mathbb{R}$, then concave functions can jump down at end points and convex functions can jump up.

Example 1. Let $C = [0, 1]$ and define

$$f(x) = \begin{cases} -x^2 & \text{if } x > 0, \\ -1 & \text{if } x = 0. \end{cases}$$

Then f is concave. It is lower semi-continuous on $[0, 1]$ and continuous on $(0, 1]$. □

3 Concavity, Convexity, and Differentiability.

A differentiable function is concave iff it lies on or below the tangent line (or plane, for $N > 1$) at any point.

Theorem 4. *Let $C \subseteq \mathbb{R}^N$ be open and convex and let $f : C \to \mathbb{R}$ be differentiable.*

1. *f is concave iff for any $x^*, x \in C$*

$$f(x) \leq Df(x^*)[x - x^*] + f(x^*).$$

2. *f is convex iff for any $x^*, x \in C$*

$$f(x) \geq Df(x^*)[x - x^*] + f(x^*).$$

Proof. If f is concave then for any $x, x^* \in C$, $x \neq x^*$, and any $\theta \in (0, 1)$, $f(\theta x + (1-\theta)x^*) \geq \theta f(x) + (1-\theta)f(x^*)$, or $f(x^* + \theta(x - x^*)) \geq f(x^*) + \theta[f(x) - f(x^*)]$ or, dividing by θ and rearranging,

$$f(x) - f(x^*) \leq \frac{f(x + \theta(x^* - x)) - f(x^*)}{\theta}.$$

Taking the limit of the right-hand side as $\theta \downarrow 0$ (giving the directional derivative of f in the direction $x^* - x$) and rearranging yields the result.[2] ■

For functions on $\mathbb{R}$ ($N = 1$), a function is concave iff its slope is decreasing, and it is convex iff its slope is increasing. One way to express this, without assuming differentiability, is the following.

Theorem 5. *Let $C \subseteq \mathbb{R}$ be an open interval.*

1. *$f : C \to \mathbb{R}$ is concave iff for any $a, b, c \in C$, with $a < b < c$,*
$$\frac{f(b) - f(a)}{b - a} \geq \frac{f(c) - f(b)}{c - b}.$$
For strictly concavity, the inequality is strict.

2. *$f : C \to \mathbb{R}$ is convex iff for any $a, b, c \in C$, with $a < b < c$,*
$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(c) - f(b)}{c - b}.$$
For strictly convexity, the inequality is strict.

Proof. Suppose that f is concave and take any $a, b, c \in C$, $a < b < c$. Since $b - a$ and $c - b > 0$, the expression above holds iff

$$[f(b) - f(a)](c - b) \geq [f(c) - f(b)](b - a),$$

which holds iff (collecting terms in $f(b)$),

$$f(b)(c - a) \geq f(a)(c - b) + f(c)(b - a),$$

which holds iff

$$f(b) \geq \left[\frac{c - b}{c - a}\right] f(a) + \left[\frac{b - a}{c - a}\right] f(c).$$

Since f is concave, the latter holds taking $\theta = (c - b)/(c - a) \in (0, 1)$ and verifying that, indeed, $b = \theta a + (1 - \theta)c$. The other claims are almost immediate. ■

For $N = 1$, Theorem 5 says that a function is concave iff its slope is decreasing. For twice differentiable functions this implies that the second derivative is negative. The following result, Theorem 6, records this fact and generalizes it to $N > 1$. Recall that a symmetric $N \times N$ matrix A is *negative semi-definite* iff, for any $v \in \mathbb{R}^N$, $v'Av \leq 0$. The matrix is *negative definite* iff, for any $v \in \mathbb{R}^N$, $v \neq 0$, $v'Av < 0$. The definitions of *positive semi-definite* and *positive definite* are analogous. As is standard practice, I write $D^2 f(x)$ for the Hessian, $D(\nabla f)(x)$, the $N \times N$ matrix of second order partial derivatives. By Young's Theorem, if f is $\mathcal{C}^2$ then $D^2 f(x)$ is symmetric.

[2] Strictly speaking, the directional derivative was defined for directions v such that $\|v\| = 1$, while it is possible that $\|x - x^*\| \neq 1$. But a slight modification of the Chain Rule argument that $D_v f(x) = Df(x)v$ establishes the result above.

Theorem 6. *Let $C \subseteq \mathbb{R}^N$ be open and convex and let $f : C \to \mathbb{R}$ be $\mathcal{C}^2$.*

1. (a) *If $D^2 f(x)$ is negative definite for every $x \in C$ then f is strictly concave.*
   (b) *If $D^2 f(x)$ is negative semi-definite for every $x \in C$ then f is concave.*
   (c) *If f is concave and $\mathcal{C}^2$ then $D^2 f(x)$ is negative semi-definite for every $x \in C$.*

2. (a) *If $D^2 f(x)$ is positive definite for every $x \in C$ then f is strictly convex.*
   (b) *If $D^2 f(x)$ is positive semi-definite for every $x \in C$ then f is convex.*
   (c) *If f is convex and $\mathcal{C}^2$ then $D^2 f(x)$ is positive semi-definite for every $x \in C$.*

Proof.

1. $N = 1$. In this case $D^2 f(x) \in \mathbb{R}$, hence $D^2 f(x)$ is negative definite iff $D^2 f(x) < 0$. For 1(a), if $D^2 f(x) < 0$ for all x then $Df(x)$ is strictly decreasing for all x. By the Mean Value Theorem, for any $a, b, c \in C$, $a < b < c$, there is an $x_a \in (a, b)$ and $x_b \in (b, c)$ such that

   $$\begin{aligned} f(b) - f(a) &= Df(x_a)(b - a), \\ f(c) - f(b) &= Df(x_b)(c - b). \end{aligned}$$

   Since $Df(x)$ is strictly decreasing, and since $x_b > x_a$, $Df(x_a) > Df(x_b)$, which implies, since $b - a > 0$ and $c - b > 0$,

   $$\frac{f(b) - f(a)}{b - a} > \frac{f(c) - f(b)}{c - b}.$$

   By Theorem 5, this implies that f is strictly concave. The proof of parts 1(b), 2(a) and 2(b) are almost identical.

   It remains to prove parts 1(c) and 2(c). The proof of part 1(c) is by contraposition. Suppose that $D^2 f(x) > 0$ for some $x^* \in C$. Since f is $\mathcal{C}^2$, $D^2 f(x^*) > 0$ for x in some ball containing x^*. Then f is strictly convex, and hence is not concave, for x in this ball. The proof of part 2(c) is almost identical.

2. $N > 1$. To show 1(a), consider any $a, b \in C$, $b \neq a$, and an open interval $\mathcal{O}$ containing $[0, 1]$ such that, for any $\theta \in \mathcal{O}$,

   $$\theta a + (1 - \theta) b \in C$$

   (such a $\mathcal{O}$ exists because C is open). Let $x_\theta = \theta a + (1 - \theta) b$.

   To show 1(a), I need to show that for any $\theta \in (0, 1)$, $f(x_\theta) > \theta f(a) + (1 - \theta) f(b)$. For any $\theta \in \mathcal{O}$, let $g(\theta) = b + \theta(a - b)$ and $h(\theta) = f(g(\theta)) = f(b + \theta(a - b))$. By the $N = 1$ step above, a sufficient condition for the strict concavity of h

on $\mathcal{O}$ is that $D^2h(\theta) < 0$ for all $\theta \in \mathcal{O}$. By the Chain Rule, for any $\theta \in \mathcal{O}$, letting $v = a - b$, $Dh(\theta) = Df(x_\theta)v$ and hence (this follows by a calculation),

$$D^2h(\theta) = v'D^2f(x_\theta)v.$$

Therefore, if $v'D^2f(x_\theta)v < 0$ for every $\theta \in \mathcal{O}$ then h is strictly concave on $\mathcal{O}$, which implies that for $\theta \in (0,1)$, since $\theta = \theta \times 1 + (1-\theta) \times 0$, $h(\theta) > \theta h(1) + (1-\theta)h(0)$, which implies $f(x_\theta) > \theta f(a) + (1-\theta)f(b)$, as was to be shown.

The proof of 1(b), 2(a), and 2(b) are similar.

For 1(c), if $D^2f(x)$ is not negative semi-definite then there is a v such that $v'D^2f(x)v > 0$ (note that I am only claiming there is at least one such v; I am not claiming that $D^2f(x)$ is positive definite). Then f is strictly convex, and hence not concave, along the line through x in the direction v. The proof of 2(c) is similar.

■

Say that f is *differentiably strictly concave* iff $D^2f(x)$ is negative definite for every x. If $N = 1$ then f is differentially strictly concave iff $D^2f(x) < 0$ for every x. The definition of *differentiable strict convexity* is analogous.

Example 2. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -x^4$. f is strictly concave but fails differentiable strict concavity since $D^2f(0) = 0$. □

4 Facts about Concave and Convex Functions.

Recall that $f : \mathbb{R}^N \to \mathbb{R}$ is affine iff it is of the form $f(x) = Ax + b$ for some $1 \times N$ matrix A and some point $b \in \mathbb{R}$. Geometrically, the graph of a real-valued affine function is a plane (a line, if the domain is $\mathbb{R}$). An important elementary fact is that real-valued affine functions are both concave and convex. This is consistent with the fact that the second derivative of any affine function is the zero matrix.

Showing that other functions are concave or convex typically requires work. For $N = 1$, Theorem 6 can be used to show that many standard functions are concave, strictly concave, and so on.

Example 3. All of the following claims can be verified with a simple calculation.

1. e^x is strictly convex on $\mathbb{R}$,

2. $\ln(x)$ is strictly concave on $\mathbb{R}_+$,

3. $1/x$ is strictly convex on $\mathbb{R}_{++}$ and strictly concave on $\mathbb{R}_{--}$,

4. x^t, where t is an integer greater than 1, is strictly convex on $\mathbb{R}_+$. On $\mathbb{R}_-$, x^t is strictly convex for t even and strictly concave for t odd.

5. x^α, where α is a real number in $(0,1)$ is strictly concave on $\mathbb{R}_+$.

□

One can often verify the concavity of other, more complicated functions by decomposing the functions into simpler pieces. The following results help do this.

Theorem 7. *Let $C \subseteq \mathbb{R}^N$ be convex. Let $f : C \to \mathbb{R}$ be concave. Let D be any interval containing $f(C)$ and let $g : D \to \mathbb{R}$ be concave and weakly increasing. Then $h : C \to \mathbb{R}$ defined by $h(x) = g(f(x))$ is concave. Moreover, if f is strictly concave and g is strictly increasing then h is strictly concave. Analogous claims hold for f convex (again with g increasing).*

Proof. Consider any $a, b \in C$ and any $\theta \in [0,1]$. Let $x_\theta = \theta a + (1-\theta)b$. Since f is concave,

$$f(x_\theta) \geq \theta f(a) + (1-\theta) f(b).$$

Then

$$\begin{aligned} h(x_\theta) = g(f(x_\theta)) &\geq g(\theta f(a) + (1-\theta) f(b)) \\ &\geq \theta g(f(a)) + (1-\theta) g(f(b)) \\ &= \theta h(a) + (1-\theta) h(b), \end{aligned}$$

where the first inequality does from the fact that f is concave and g is weakly increasing and the second inequality comes from the fact that g is concave. The other parts of the proof are essentially identical. ■

Example 4. Let the domain be $\mathbb{R}$. Consider $h(x) = e^{1/x}$. Let $f(x) = 1/x$ and let $g(y) = e^y$. Then $h(x) = g(f(x))$. Function f is strictly convex and g is (strictly) convex and strictly increasing. Therefore, by Theorem 7, h is strictly convex. □

It is important in Theorem 7 that g be increasing.

Example 5. Let the domain by $\mathbb{R}$. Consider $h(x) = e^{-x^2}$. This is just the standard normal density except that it is off by a factor of $1/\sqrt{2\pi}$. Let $f(x) = e^{x^2}$ and let $g(y) = 1/y$. Then $h(x) = g(f(x))$. Now, f is convex on $\mathbb{R}_{++}$ (indeed, on $\mathbb{R}$) and g is also convex on $\mathbb{R}_{++}$. The function h is not, however, convex. While it is strictly convex for $|x|$ sufficiently large, for x near zero it is strictly concave. This does not contradict Theorem 7 because g here is decreasing. □

Theorem 8. *Let $C \subseteq \mathbb{R}$ be an interval and let $f : C \to \mathbb{R}$ be concave and strictly positive for all $x \in C$. Then $h : C \to \mathbb{R}$ defined by $h(x) = 1/f(x)$ is convex.*

Proof. Follows from Theorem 7, the fact that if f is concave then $-f$ is convex, and the fact that the function $g(x) = -1/x$ is convex and increasing on $\mathbb{R}_-$. ■

Theorem 9. *Let $C \subseteq \mathbb{R}$ be an open interval. If $f : C \to \mathbb{R}$ is strictly increasing or decreasing then $f^{-1} : f(C) \to C$ is well defined. If, in addition, f is concave or convex, then $f(C)$ is convex and the following hold.*

1. *If f is concave and strictly increasing then f^{-1} is convex.*
2. *If f is concave and strictly decreasing then f^{-1} is concave.*
3. *If f is convex and strictly increasing then f^{-1} is concave.*
4. *If f is convex and strictly decreasing then f^{-1} is convex.*

Proof. For 1, by Theorem 3, f is continuous. It follows (see the notes on connected sets) that $f(C)$ is an interval, and hence is convex. Let y, $\hat{y}$ be any two points in $f(C)$ and let $x = f^{-1}(y)$, $\hat{x} = f^{-1}(\hat{y})$. Take any $\theta \in [0,1]$. Then, since f is concave,

$$\begin{aligned} f(\theta x + (1-\theta)\hat{x}) &\geq \theta f(x) + (1-\theta) f(\hat{x}) \\ &= \theta y + (1-\theta)\hat{y}. \end{aligned}$$

Taking the inverse of both sides yields, since f is strictly increasing, and since $x = f^{-1}(y)$, $\hat{x} = f^{-1}(\hat{y})$,

$$\begin{aligned} \theta f^{-1}(y) + (1-\theta) f^{-1}(\hat{y}) &= \theta x + (1-\theta)\hat{x} \\ &\geq f^{-1}(\theta y + (1-\theta)\hat{y}). \end{aligned}$$

Since y, $\hat{h}$, and θ were arbitrary, this implies that f^{-1} is convex. The other results are similar. Note in particular that if f is concave but strictly decreasing then the last inequality above flips, and f^{-1} is concave. ■

Example 6. Let $C = (0, \infty)$ and let $f(x) = \ln(x)$. This function is (strictly) concave and strictly increasing. Its inverse is $f^{-1}(y) = e^y$, which is (strictly) convex and strictly increasing. □

Example 7. Let $C = (-\infty, 0)$ and let $f(x) = \ln(-x)$. This function is (strictly) concave and strictly decreasing. Its inverse is $f^{-1}(y) = -e^y$, which is (strictly) concave and strictly decreasing. □

Theorem 10. *Let $C \subseteq \mathbb{R}^N$ be convex. Let $f_1 : C \to \mathbb{R}$ and $f_2 : C \to \mathbb{R}$ be concave.*

1. *The function $f_1 + f_2$ is concave. Moreover, if either f_1 or f_2 is strictly concave then $f_1 + f_2$ is strictly concave.*
2. *For any $r \geq 0$, the function rf_1 is concave. Moreover, if f_1 is strictly concave then for any $r > 0$, rf_1 is strictly concave.*

Analogous claims hold if f_1, f_2 are convex.

Proof. Omitted. Almost immediate from the definition of concavity. ■

A special case in which $N > 1$ is effectively as easy to analyze as $N = 1$ is when the function is separable in the sense that is the sum of univariate functions. For example, the function $h(x_1, x_2) = e^{x_1} + e^{x_2}$ is separable. More generally, suppose $C_n \subseteq \mathbb{R}$ are intervals, let $f_n : C \to \mathbb{R}$ be twice differentiable, let $C = \prod_n C_n$, and let $g : C \to \mathbb{R}$ be defined by

$$h(x) = \sum_n f_n(x_n).$$

Then $D^2 f(x^*)$ is a diagonal matrix,

$$D^2 h(x^*) = \begin{bmatrix} D^2 f_1(x^*) & 0 & \cdots & 0 & 0 \\ 0 & D^2 f_2(x^*) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D^2 f_{N-1}(x^*) & 0 \\ 0 & 0 & \cdots & 0 & D^2 f_N(x^*) \end{bmatrix}.$$

This matrix is negative semi-definite iff the diagonal terms are all less than or equal to zero; it is negative definite iff the diagonal terms are all strictly negative.

This observation about separable functions is superficially similar to Theorem 10 but there are important differences, as illustrated in the following example.

Example 8. Consider $h(x_1, x_2) = e^{x_1} + e^{x_2}$. Since h is separable, the above observation, and the fact that the second derivative of e^x is always strictly positive, implies that h is strictly convex.

I can also employ Theorem 10, but I reach the weaker conclusion that h is convex, rather than strictly convex. Explicitly, I can view e^{x_1} not as a function on $\mathbb{R}$ but as a function on $\mathbb{R}^2$ where the second coordinate is simply ignored. Viewed as a function on $\mathbb{R}^2$, e^{x_1} is convex (one can use Theorem 7 to show this) but not strictly convex. Because the e^{x_n} are convex but not strictly convex, Theorem 10 implies only that h is convex, not strictly convex. □

References

Rockafellar, T. (1970): *Convex Analysis.* Princeton University Press, Princeton, NJ.