Econ 511
Professor: John Nachbar
October 5, 2008

Continuity and Connectedness

Theorem 1. *Let (X, d_x) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be continuous. Then for any connected set $E \subseteq X$, $f(E)$ is connected.*

Proof. I argue by contraposition. Suppose that $f(E)$ is separated. I must show that this implies that E is separated.

Let $F = f(E)$. If F is separated then there are open sets V_1, V_2 such that, $V_1 \cap F \neq \emptyset$, $V_2 \cap F \neq \emptyset$, $F \subseteq V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$. Let $O_1 = f^{-1}(V_1)$ and $O_2 = f^{-1}(V_2)$. By continuity, O_1 and O_2 are open. Since, $V_1 \cap V_2 = \emptyset$, $O_1 \cap O_2 = \emptyset$. Since $F \subseteq V_1 \cup V_2$, $E \subseteq O_1 \cup O_2$. Finally, since $V_1 \cap F \neq \emptyset$, $O_i \cap E \neq \emptyset$, for $i \in \{1, 2\}$. ■

The above theorem says that connectedness is a topological property, meaning a property that is preserved by transformation by a continuous function. Compactness is another important example of a topological property. In contrast, convexity is *not* a topological property.

Theorem 2 (Intermediate Value Theorem)**.** *Let $[a, b] \subseteq \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ then, for any $y \in (f(a), f(b))$, there is an $x \in (a, b)$ such that $f(x) = y$. And an analogous result holds if $f(b) < f(a)$.*

Proof. $f([a, b])$ is an interval, since it is connected and since any connected set in $\mathbb{R}$ is an interval. Suppose $f(a) < f(b)$. Since $f(a) \in f([a, b])$ and $f(b) \in f([a, b])$, it follows that $[f(a), f(b)] \subseteq f([a, b])$. So, if $y \in (f(a), f(b))$ then $y \in f([a, b])$ and so there is an $x \in [a, b]$ such that $f(x) = y$. Since $y \neq f(a)$ and $y \neq f(b)$, $x \in (a, b)$. The argument for $f(b) < f(a)$ is similar. ■

Example 1. Suppose $f(a) < 0$ and $f(b) > 0$. Then there is an $x \in (a, b)$ such that $f(x) = 0$.

This fact can be used to show the existence of a competitive equilibrium if there are only 2 commodities. To handle the case of more than 2 commodities, more sophisticated machinery must be employed (in particular, the Brouwer fixed point theorem). □