Econ 511
Professor: John Nachbar
October 25, 2010

Continuity

1 Definitions and Basic Theorems

Throughout these notes, (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces.

Theorem 1. *Consider any $f : X \to Y$. The following are equivalent.*

1. *For any sequence $\{x_t\}$ and point x^* in X, if $x_t \to x^*$ then $f(x_t) \to f(x^*)$.*
2. *For any $x^* \in X$ and any $\varepsilon > 0$, there is a $\delta > 0$ such that, setting $y^* = f(x^*)$, if $x \in N_\delta(x^*)$ then $f(x) \in N_\varepsilon(y^*)$; more concisely,*

$$N_\delta(x^*) \subseteq f^{-1}(N_\varepsilon(y^*)).$$

3. *For any open set $V \subseteq Y$, $f^{-1}(V)$ is open.*

Proof.

- $1 \Rightarrow 2$. By contraposition. Suppose that there is an x^* and an $\varepsilon > 0$ such that for any $\delta > 0$, setting $y^* = f(x^*)$, there is an $x \in N_\delta(x^*)$ such that $f(x) \notin N_\varepsilon(y^*)$. Construct a sequence of such x with $x_t \in N_{1/t}(x^*)$. Then, by construction, $x_t \to x^*$ but $f(x_t) \not\to y^*$.
- $2 \Rightarrow 3$. Let $V \subseteq Y$ be open. To show that $f^{-1}(V)$ is open, I show that every point in $f^{-1}(V)$ is interior to $f^{-1}(V)$. Let $x^* \in f^{-1}(V)$ (if $f^{-1}(V) = \emptyset$ then it is immediate that $f^{-1}(V)$ is open). Since V is open and $y^* = f(x^*) \in V$, y^* is interior to V, hence there is a $\varepsilon > 0$ such that $N_\varepsilon(y^*) \subseteq V$. By condition 2, there is then a $\delta > 0$ such that $N_\delta(x^*) \subseteq f^{-1}(N_\varepsilon(y^*))$. Since $f^{-1}(N_\varepsilon(y^*)) \subseteq f^{-1}(V)$, it follows that x^* is interior to $f^{-1}(V)$, as was to be shown.
- $3 \Rightarrow 1$. Suppose that $x_t \to x^*$ and set $y^* = f(x^*)$. Since $N_\varepsilon(y^*)$ is open, condition 3 implies that $f^{-1}(N_\varepsilon(y^*))$ is open, hence x^* is interior to $f^{-1}(N_\varepsilon(y^*))$, hence there is a $\delta > 0$ such that $N_\delta(x) \subseteq f^{-1}(N_\varepsilon(y))$ (this shows that condition $3 \Rightarrow 2$). Since $x_t \to x^*$, there is a T such that for all $t > T$, $x_t \in N_\delta(x^*)$, hence $f(x_t) \in N_\varepsilon(y^*)$. This shows that $f(x_t) \to y^*$.

■

It is standard in Topology to take the last of the above characterizations as the definition of continuity.

Definition 1. *$f : X \to Y$ is* continuous *iff for every open $V \subseteq Y$, $f^{-1}(V)$ is open.*

The definition does *not* imply that if $U \subseteq X$ is open then $f(U)$ is open.

Theorem 2. *$f : X \to Y$ is continuous iff for every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed.*

Proof. $\Rightarrow$ Let B be closed. Then B^c is open. Since f is continuous, $f^{-1}(B^c)$ is open.

I claim that $f^{-1}(B^c) = (f^{-1}(B))^c$. This follows from the fact that $x \in f^{-1}(B^c)$ iff $f(x) \in B^c$ iff $f(x) \notin B$ iff $x \notin f^{-1}(B)$ iff $x \in (f^{-1}(B))^c$.

Since $f^{-1}(B^c) = (f^{-1}(B))^c$ and $f^{-1}(B^c)$ is open, $(f^{-1}(B))^c$ is open, hence $f^{-1}(B)$ is closed, as was to be shown.

$\Leftarrow$ Similar and therefore omitted. ■

Functions that are not continuous may still be continuous at certain points, in the following sense.

Definition 2. *$f : X \to Y$ is* continuous at $x^* \in X$ *iff, for any open set $V \subseteq Y$ such that $f(x^*) \in V$, $f^{-1}(V)$ is open.*

In view of Theorem **??**, f is continuous at x^* iff either of the following conditions hold. Set $y^* = f(x^*)$.

1. For any $\{x_t\}$ converging to x^*, $f(x_t) \to y^*$.
2. For any $\varepsilon > 0$ there is a $\delta > 0$ such that $N_\delta(x^*) \subseteq f^{-1}(N_\varepsilon(y^*))$.

It is immediate that f is continuous iff it is continuous at every point $x \in X$.

Example 1. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. I claim that f is continuous at $x^* = 0$. For any $\varepsilon > 0$, $f^{-1}(-\varepsilon, \varepsilon) = (-\sqrt[3]{\varepsilon}, \sqrt[3]{\varepsilon})$. So set $\delta = \sqrt[3]{\varepsilon}$. Note that δ is a function of ε. The continuity of f (at every point) can be verified in this way, but it also follows from Theorem **??** below. □

Example 2. Consider $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

This is not continuous.. Take $x^* = 0$ and $\varepsilon = 1/2$. Then $f^{-1}(N_{1/2}(0)) = (-\infty, 0]$, which is not open. □

Finally, it is standard practice to rewrite condition 1 in Theorem **??** in a form that does not refer explicitly to a particular sequence.

Definition 3. *Let $A \subseteq X$, let $f : A \to Y$, and let $x^* \in X$ be a limit point of A. Write*

$$\lim_{x \to x^*} f(x) = y^*$$

or

$$f(x) \to y^* \text{ as } x \to x^*$$

iff $f(x_t) \to y^*$ for every sequence $\{x_t\}$ in X such that $x_t \neq x^*$ for all t and $x_t \to x^*$.

If $Y = \mathbb{R}$, write

$$\lim_{x \to x^*} f(x) = \infty$$

iff $f(x_t) \to \infty$ for every sequence $\{x_t\}$ in X such that $x_t \neq x^*$ for all t and $x_t \to x^*$. Similarly, write $\lim_{x \to x^*} f(x) = -\infty$ iff $f(x_t) \to -\infty$ for every sequence $\{x_t\}$ in X such that $x_t \neq x^*$ for all t and $x_t \to x^*$.

Remark 1. Note that f does *not* need to be defined at x^* in order to talk about $\lim_{x \to x^*} f(x)$. □

It is then immediate from condition 1 in Theorem **??** that $f : X \to Y$ is continuous at $x^* \in X$ iff, setting $y^* = f(x^*)$, $\lim_{x \to x^*} f(x) = y^*$.

Example 3. Let $f : \mathbb{R}_{++} \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then $f(x) \to +\infty$ as $x \to 0$. □

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

There is no limit at 0. □

Example 5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then f is not continuous at 0: $\lim_{x \to 0} f(x) = 0 \neq 1 = f(0)$. □

2 Some Facts about Continuous Functions

Theorem 3. *Fix $y^* \in Y$ and let $f : X \to Y$ be the constant function defined by, for all $x \in X$, $f(x) = y^*$. Then f is continuous.*

Proof. For any open $B \subseteq Y$, if $y^* \in B$ then $f^{-1}(B) = X$, which is open, while if $y^* \notin B$ then $f^{-1}(B) = \emptyset$, which is also open. ■

Theorem 4. *Let $f : X \to X$ be the* identity*: for all $x \in X$, $f(x) = x$. Then f is continuous.*

Proof. Consider any open set $A \subseteq X$. Then, since f is the identity, $f^{-1}(A) = A$, which is open. ■

Theorem 5. *Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$. Then if f and g are continuous at x^*, so is $f + g$, fg and f/g (provided $g(x) \neq 0$ for all x in some neighborhood of x^*).*

Proof. Immediate from Theorem 7 of the notes on Sequences and Series. ■

Thus the sum, product, and ratio of continuous real-valued functions are continuous.

Theorem 6. *Any polynomial is continuous.*

Proof. Follows by induction from Theorem **??**, Theorem **??**, and Theorem **??**. ■

Theorem 7. *Let $f : X \to \mathbb{R}^N$. Let f_ℓ be the ℓ coordinate function of f; $f(x) = (f_1(x), \ldots, f_\ell(x), \ldots, f_L(x))$. Then f continuous at x^* iff f_ℓ is continuous at x^* for each ℓ.*

Proof. Follows from Theorem 8 of the notes on Sequences and Series. ■

Theorem 8. *Let $f : X \to \mathbb{R}^N$, $g : X \to \mathbb{R}^N$ be continuous at x^*. Then $f + g$ and $f \cdot g$ are continuous at x^*.*

Proof. Follows from Theorem 9 of the notes on Sequences and Series. ■

Theorem 9. *Let $f : X \to Y$ and $g : Y \to Z$. If f is continuous at x^* and g is continuous at $f(x^*)$ then $g \circ f$ is continuous at x^*.*

Proof. Let $y^* = f(x^*)$ and let $z^* = g(y^*)$. Consider any $\varepsilon > 0$. Since g is continuous, there is a $\gamma > 0$ such that $N_\gamma(y^*) \subseteq g^{-1}(N_\varepsilon(z^*))$. Since f is continuous, there is a $\delta > 0$ such that $N_\delta(x^*) \subseteq f^{-1}(N_\gamma(y^*))$. Thus, $N_\delta(x^*) \subseteq (g \circ f)^{-1}(N_\varepsilon(z^*))$, as was to be shown. ■