Econ 511
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Connected Sets

1 Definition.

Definition 1. *Let (X, d) be a metric space. A non-empty set $E \subseteq X$ is* separated *iff there are nonempty sets open sets O_1 and O_2 such that $O_1 \cap E \neq \emptyset$, $O_2 \cap E \neq \emptyset$, $E \subseteq O_1 \cup O_2$, but $O_1 \cap O_2 = \emptyset$. A set that is not separated is* connected.

On the real line, connected sets can only be of a certain form. A set $E \subseteq \mathbb{R}$ is an *interval* iff for any $a, b \in E$ and any x such that $a < x < b$, $x \in E$. An equivalent characterization is that $E \subseteq \mathbb{R}$ is an interval iff it is *convex*: for any $a, b \in E$, $a \neq b$, and any $\theta \in (0, 1)$, $\theta a + (1 - \theta)b \in E$. To the see the equivalence, note that for any a, b, with $a < b$, x satisfies $a < x < b$ iff $x = \theta a + (1 - \theta)b$, with $\theta = (b - x)/(b - a)$.

Theorem 1. *Let E be a subset of $\mathbb{R}$. E is connected iff it is an interval.*

Proof. By contraposition.

$\Leftarrow$. Suppose that E is not connected. Then there are non-empty open sets O_1 and O_2 such that $O_1 \cap E \neq \emptyset$, $O_2 \cap E \neq \emptyset$, $E \subseteq O_1 \cup O_2$, but $O_1 \cap O_2 = \emptyset$. Choose any $a \in O_1 \cap E$, $b \in O_2 \cap E$. Without loss of generality, suppose $a < b$. Let $E_b = \{x \in O_1 : x < b\}$. E_b is bounded above (by b) and non-empty (since $a \in E_b$). Let $x^* = \sup E_b$.

I claim that $a < x^* < b$ and $x^* \notin O_1 \cup O_2$, hence $x^* \notin E$, hence E is not an interval.

1. $a < x^*$. Since $a \in O_1$ and O_1 is open, there is an $\varepsilon > 0$ such that $N_\varepsilon(a) \subseteq O_1$. Since $a < b$ and $b \notin O_1$, this implies that $x + \varepsilon \leq b$, hence $N_\varepsilon(a) \subseteq E_b$, hence x^* is an upper bound for $N_\varepsilon(a)$, hence $a + \varepsilon \leq x^*$, hence $a < x^*$.

2. $x^* < b$. Conversely, since $b \in O_2$ and O_2 is open, there is an $\varepsilon > 0$ such that $N_\varepsilon(b) \subseteq O_2$. Therefore, if $x \in E_b$, then $x \leq b - \varepsilon$. Therefore, $x^* \leq b - \varepsilon$, hence $x^* < b$.

3. $x^* \notin O_1 \cup O_2$. By contraposition. Take any $x < b$. If $x \in O_1$, then, since O_1 is open, there is an $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq O_1$. Since $x < b$ and $b \notin O_1$, this implies that $N_\varepsilon(x) \subseteq E_b$, hence $x + \varepsilon \leq x^*$, hence $x < x^*$. Since $x^* < b$ but is strictly greater than any $x \in O_1$ that is less than b, $x^* \notin O_1$. On the other hand, if $x \in O_2$, then, since O_2 is open, there is an $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq O_2$. If x is an upper bound for E_b, then this implies that $x - \varepsilon$ is also an upper bound for E_b, hence $x^* \leq x - \varepsilon$, hence $x^* < x$. Since x^* is an upper bound of E_b but is strictly less than any upper bound that is in O_2, $x^* \notin O_2$.

$\Rightarrow$. Suppose that E is not an interval. Then there is are points $a, b \in E$, with $a < b$, and a number x^* such that $a < x^* < b$ and $x^* \notin E$. Let $O_1 = (-\infty, x^*)$ and $O_2 = (x^*, \infty)$. Then $a \in O_1$, hence $O_1 \cap E \neq \emptyset$, $b \in O_2$, hence $O_2 \cap E \neq \emptyset$, $E \subseteq O_1 \cup O_2$ but $O_1 \cap O_2 = \emptyset$, as was to be shown. ■

If I remove the "middle" from interval in $\mathbb{R}$, I get a separated set: although $[0, 3]$ is connected, $[0, 1) \cup (2, 3]$ is separated. This is not true in higher dimensions. This is an important topological difference between $\mathbb{R}$ and Euclidean spaces of higher dimension. It is the fundamental reason why some useful results for $\mathbb{R}$, notably the Mean Value Theorem, do not fully generalize, and why other useful results, such as the Intermediate Value Theorem, generalize only with great difficulty (here, I am interpreting the Brouwer Fixed Point Theorem as the generalization of the Intermediate Value Theorem).