Econ 4111
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Compactness and Completeness in $\mathbb{R}^\infty$.

1 Metrics for $\mathbb{R}^\infty$

There is no default metric for $\mathbb{R}^\infty$. Instead, I define two standard metrics, each of which can be useful, depending on the application.

1.1 The d_{sup} metric.

The sup metric is the analog in $\mathbb{R}^\infty$ of $d_{\max}$ in $\mathbb{R}^N$. Given $a, b \in \mathbb{R}^\infty$,

$$d_{\text{sup}}(a, b) = \sup_n \{|a_n - b_n|\}.$$

Note that this definition has a problem: if $a = (1, 2, 3, \dots)$ then $d_{\text{sup}}(a, 0)$ is either undefined or infinite, take your pick.

There are two standard ways to address this problem. One way is to truncate the coordinate by coordinate measure of distance to a maximum value of, say, 1. That is, instead of $\sup_n\{|a_n - b_n|\}$, consider $\sup_n\{\min\{1, |a_n - b_n|\}\}$. This modification is effective but in practice it is not much used.

Instead, it is more common to restict attention to points in $\mathbb{R}^\infty$ for which d_{sup}, as originally defined, makes sense. These are the points x that are bounded in the sense that $\sup_n |x_n| < \infty$. The set of such points is denoted ℓ^∞.

For example, if $a = (\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots)$ and $b = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots)$ then $a - b = (0, \frac{1}{2}, \frac{3}{4}, \dots)$ and so $d_{\text{sup}}(a, b) = 1$.

As usual, to be careful, I have to show that d_{sup} is a metric.

Theorem 1. *The metric d_{sup} on ℓ^∞ satisfies the three metric properties.*

Proof. The proof is almost identical to the proof for $d_{\max}$ in $\mathbb{R}^N$, so I omit it. ■

An important fact about the d_{sup} metric is that convergence under the d_{sup} metric is more demanding than pointwise convergence in the sense that every sequence that converges under d_{sup} converges pointwise but not necessarily conversely. This is in sharp contrast to the situation with $d_{\max}$ in $\mathbb{R}^N$, where convergence was equivalent to pointwise convergence.

Theorem 2. *In ℓ^∞, convergence under d_{sup} implies pointwise convergence but not conversely.*

Proof. The proof that convergence under d_{sup} implies pointwise convergence is essentially the same as for d_{max} in $\mathbb{R}^{\mathbb{N}}$, so I omit it.

To prove that pointwise convergence does not imply convergence under d_{sup}, it suffices to give an example. Consider the sequence $\{x_t\}$ where

$$x_t = (0, \dots, 0, 1, 0, \dots),$$

with the 1 appearing in coordinate $n = t$. Then $\{x_t\}$ converges pointwise to the origin; indeed, for every coordinate n, that coordinate is 0 in every period starting in period $t = n + 1$. But this sequence does not converge in d_{sup}. Indeed, for every t, $d_{\text{sup}}(x_t, 0) = 1$. ■

1.2 The d_p metric.

As just noted, for the metric d_{sup}, convergence is not equivalent to pointwise convergence. I now give a metric for which convergence *is* equivalent to pointwise convergence. For any $a, b \in \mathbb{R}^\infty$, define

$$d_p(a, b) = \sup_n \left\{ \frac{\min\{1, |a_n - b_n|\}}{n} \right\}.$$

Loosely, the difference between d_p and d_{sup} is that d_p weights high-numbered coordinates less than low numbered coordinates. In addition, notice that I have truncated the coordinate by coordinate measure of distance to a maximum value of 1, which ensures that d_p is defined for all of $\mathbb{R}^\infty$ rather than just for a subset like ℓ^∞. This truncation leads to the odd feature that, in this metric, $N_2(0) = \mathbb{R}^\infty$, which means that all of $\mathbb{R}^\infty$ is bounded! This strange feature notwithstanding, d_p is a valid metric.

Theorem 3. *The metric d_p on $\mathbb{R}^\infty$ satisfies the three metric properties.*

Proof. The proof is again very similar to that for d_{max} in $\mathbb{R}^N$, so I omit it. ■

Theorem 4. *In the metric space $(\mathbb{R}^\infty, d_p)$, convergence is equivalent to pointwise convergence.*

Proof. To see that convergence implies pointwise convergence, consider any sequence $\{x_t\}$ and suppose that $x_t \to x$ under d_p. Fix any $\varepsilon > 0$, $\varepsilon < 1$ and any coordinate n. Then there is a T such that for all $t > T$, $d_p(x_t, x) < \varepsilon/n$ which implies that, in particular,

$$\frac{\min\{1, |x_{tn} - x_n|\}}{n} < \frac{\varepsilon}{n}$$

which implies $|x_{tn} - x_n| < \varepsilon$, as was to be shown.

To see that pointwise convergence implies convergence, suppose that $x_t \to x$ pointwise. Fix ε and choose N sufficiently large that $1/N < \varepsilon$. Looking just at the first N coordinates, let $\hat{x}_t = (x_{t1}, \ldots, x_{tN})$ and let $\hat{x} = (x_1, \ldots, x_N)$. Then the sequence $\{\hat{x}_t\}$ converges pointwise to $\hat{x}$. In $\mathbb{R}^N$, if $\{\hat{x}_t\} \to \hat{x}$ pointwise then it converges under $d_{\max}$. Therefore, there is a T such that for all $t > T$ and all $n \leq N$, $|x_{tn} - x_n| < \varepsilon$. By the construction of N, it follows that for all n,

$$\frac{\min\{1, |x_{tn} - x_n|\}}{n} < \varepsilon,$$

which implies $d_p(x_t, x) < \varepsilon$, as was to be shown. ■

It follows from Theorem 2 and Theorem 4 that any sequence (in ℓ^∞) that converges under $d_{\sup}$ converges under d_p, but that there are sequences that converge under d_p but not under $d_{\sup}$.

There are other metrics on $\mathbb{R}^\infty$, many other metrics, in fact, for which convergence is equivalent to pointwise convergence. The metric d_p is simply one that is tractable and commonly used.

2 Completeness in $\mathbb{R}^\infty$.

Theorem 5. *Both* $(\ell^\infty, d_{\sup})$ *and* $(\mathbb{R}^\infty, d_p)$ *are complete.*

Proof.

1. $(\ell^\infty, d_{\sup})$. Fix any Cauchy sequence $\{x_t\}$ in ℓ^∞. As was the case in $\mathbb{R}^N$, if $\{x_t\}$ is Cauchy under $d_{\sup}$ then for each n each coordinate sequence $\{x_{tn}\}$ is Cauchy. Since $\mathbb{R}$ is complete, for each n there is an x^*_n such that $x_{tn} \to x^*_n$. That is, $\{x_t\}$ converges pointwise to x^*. I claim that $x_t \to x^*$. In this metric space, it is not always true that pointwise convergence implies convergence (see Theorem 2). But the situation here is special because $\{x_t\}$ is Cauchy.

   Consider any $\varepsilon > 0$ and choose T such that for all $t, s > T$, $d_{\sup}(x_t, x_s) < \varepsilon/4$. I claim that for any $t > T$, $d_{\sup}(x_t, x^*) < \varepsilon$.

   Fix $t > T$. For each n, since $\{x_{sn}\}$ converges to x^*_n, there is a T_n such that $|x_{sn} - x^*_n| < \varepsilon/4$ for all $s > T_n$. Choose any $s > \max\{T, T_n\}$. Then

   $$|x_{tn} - x^*_n| \leq |x_{tn} - x_{sn}| + |x_{sn} - x^*_n| < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.$$

   Since this holds for all n, $d_{\sup}(x_t, x^*) \leq \varepsilon/2 < \varepsilon$, as was to be shown.

   Finally, to see that $x^* \in \ell^\infty$, note that for any n and t, $|x^*_n| \leq |x^*_n - x_{tn}| + |x_{tn}| \leq \sup_n [|x^*_n - x_{tn}| + |x_{tn}|] \leq \sup_n |x^*_n - x_{tn}| + \sup_n |x_{tn}| = d_{\sup}(x^*, x_t) + d_{\sup}(x_t, 0)$. Take t large enough that $d_{\sup}(x^*, x_t) < 1$. Then for all n, $|x^*_n| < d_{\sup}(x_t, 0) + 1$, hence $\sup_n |x^*_n| < d_{\sup}(x_t, 0) + 1 < \infty$, as was to be shown.

2. $(\mathbb{R}^\infty, d_p)$. The proof is essentially the same as the proof for $\mathbb{R}^N$. More explicitly, a modification of the proof of Theorem 4 shows that if $\{x_t\}$ is Cauchy under d_p then for each n, $\{x_{tn}\}$ is Cauchy. As in the proof of the first part of the theorem, this implies that $\{x_t\}$ converges to x^*_t pointwise. By Theorem 4, this implies $x_t \to x^*$.

■

3 Compactness in $\mathbb{R}^\infty$.

Since both $(\ell^\infty, d_{\text{sup}})$ and $(\mathbb{R}^\infty, d_p)$ are complete, a set will be compact iff it is closed and totally bounded (all with respect to the relevant metric). One would like to replace totally bounded with bounded, as was the case in $\mathbb{R}^N$, but in either of these two metric spaces there exist bounded sets that are not totally bounded. Hence there exist closed and bounded sets that are not compact.

Theorem 6. *In both* $(\ell^\infty, d_{\text{sup}})$ *and* $(\mathbb{R}^\infty, d_p)$*, there exist bounded sets that are not totally bounded.*

Proof.

1. $(\ell^\infty, d_{\text{sup}})$. The ball $N_2(0)$ is bounded (trivially) but not totally bounded. To see that it is not totally bounded, consider points of the form $(1, 0, \dots)$, $(0, 1, 0, \dots)$, and so on. The distance between any two of these points is 1, so at most one of them can be in any ball of radius 1/2 (and hence diameter 1). There are infinitely many of these points, so there is no finite set S such that $N_2(0) \subseteq \bigcup_{x \in S} N_{1/2}(x)$.
2. $(\mathbb{R}^\infty, d_p)$. As noted earlier, the metric d_p has the property that $N_2(0) = \mathbb{R}^\infty$. Thus $\mathbb{R}^\infty$ is bounded. It is not, however, totally bounded. Indeed, points of the form $(1, 0, 0 \dots)$, $(2, 0, 0 \dots)$ are all of distance 1 from each other. The argument is then the same as the one above.

■

Morever, the unit cube – an extremely nice set by most standards – is not compact in $(\ell^\infty, d_{\text{sup}})$.

Theorem 7. *Let* $S = \{x \in \ell^\infty : x_n \in [0, 1] \ \forall n\}$. *S is not compact in* $(\ell^\infty, d_{\text{sup}})$.

Proof. The proof of Theorem 6 used only points in S and therefore shows that S is not totally bounded. ■

The unit cube *is* compact, however, in $(\mathbb{R}^\infty, d_p)$. This is a special case of the following more general result.

Theorem 8 ($\mathbb{R}^\infty$ Tychonoff)**.** *For each n, let $C_n \subset \mathbb{R}$ be compact. Then*

$$C = \{x \in \mathbb{R}^\infty : x_n \in C_n \ \forall n\}$$

is compact in $(\mathbb{R}^\infty, d_p)$.

Proof. It suffices to show that C is closed and totally bounded. The proof of the analogous result for $\mathbb{R}^N$ is easily adapted to show that C is closed.

To see that C is totally bounded, consider any $\varepsilon > 0$ and choose N sufficiently large that $1/N < \varepsilon$. Let $\hat{C} = \{\hat{x} \in \mathbb{R}^N : \hat{x}_n \in C_n \text{ for } n \in \{1, \dots, N\}\}$. Then $\hat{C}$ is compact in $\mathbb{R}^N$, hence totally bounded. Therefore, there is a finite set $\hat{S} \subset \mathbb{R}^N$ such that the balls $N_\varepsilon(\hat{x})$, for $\hat{x} \in \hat{S}$, cover $\hat{C}$, where $N_\varepsilon(\hat{x})$ is defined with respect to d_E.

For each $\hat{x} \in \hat{S}$, let $x = (\hat{x}_1, \dots, \hat{x}_N, 0, \dots, 0)$. Let S be the set of these x; S is finite since $\hat{S}$ is. Consider any point $a \in C$. Let $\hat{a} = (a_1, \dots, a_N)$. Then $\hat{a} \in \hat{C}$ and hence there is an $\hat{x} \in \hat{S}$ such that $\hat{a} \in N_\varepsilon(\hat{x})$, again with $N_\varepsilon(\hat{x})$ defined with respect to d_E. That is, $d_E(\hat{a}, \hat{x}) < \varepsilon$. It follows that $d_{\max}(\hat{a}, \hat{x}) < \varepsilon$, hence $\max_{n \leq N} |a_n - x_n| < \varepsilon$, hence

$$\max_{n \leq N} \frac{\min\{1, |a_n - x_n|\}}{n} < \varepsilon.$$

On the other hand, by the construction of N, for any $n > N$,

$$\frac{\min\{1, |a_n - x_n|\}}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore, $d_p(a, x) < \varepsilon$, hence $a \in N_\varepsilon(x)$, where $N_\varepsilon(x)$ is defined with respect to d_p. In summary, for any $a \in C$ there is an $x \in S$ such that $a \in N_\varepsilon(x)$, which shows that C is totally bounded. ■

"$\mathbb{R}^\infty$ Tychonoff" is a special case of a much more general result, Tychonoff's theorem, that says that in product spaces, the product of compact sets is compact under pointwise convergence.

Theorem 7 and Theorem 8 illustrate the following general fact.

Theorem 9. *Any set in ℓ^∞ that is compact under $d_{\sup}$ is also compact under d_p. There are, however, sets in ℓ^∞ that are compact under d_p but not under $d_{\sup}$.*

Proof. To show that any set in ℓ^∞ that is compact under $d_{\sup}$ is also compact under d_p, it suffices to argue that if a sequence converges under $d_{\sup}$ then it converges under d_p. But this is implied by Theorem 2 and Theorem 4. The second part of the theorem follows from Theorem 7 and Theorem 8. ■

Sets that *are* compact in $(\ell^\infty, d_{\sup})$ are closed rectangles such that the length of side n is less than or equal to $1/n$. More generally, it is necessary and sufficient that for each $\varepsilon > 0$ there are only finitely many sides of length greater than ε. The proof is similar to that of Theorem 8.