Econ 4111
Professor: John Nachbar
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Compactness

1 Introduction.

An important fact about metric spaces is that the following five properties of a set C are equivalent.

1. C is *compact*: for any set $\mathcal{O}$ of open sets with the property that
$$C \subseteq \bigcup_{O \in \mathcal{O}} O$$
there is a finite subset $\hat{\mathcal{O}} \subseteq \mathcal{O}$ such that
$$C \subseteq \bigcup_{O \in \hat{\mathcal{O}}} O.$$

2. For any set $\mathcal{A}$ of closed sets in C, if every finite subset of $\mathcal{A}$ has a non-empty intersection then all of $\mathcal{A}$ has a non-empty intersection.

3. C is *sequentially compact*: every sequence in C has a subsequence that converges to a point in C.

4. C is *limit point compact*: every infinite subset of C has a limit point in C.

5. C is complete and totally bounded.

Of these properties, 1 and 2 are somewhat hard to grasp and so I will not discuss them in detail. For us, the essential properties will be 3 and 5. These two properties have different uses. Property 5 is used to identify compact sets while property 3 is used to prove facts about compact sets. In particular, I use 3 to show existence of a solution to optimization problems. I prove the equivalence of 3 and 5 below.

Although I will not prove the equivalence of 1 and 3, I take advantage of this equivalence and refer to sets as being compact rather than sequentially compact. Proving that 1 implies 3 is not difficult but proving that 3 implies 1 takes some work. (The fact that 1 and 2 are equivalent is almost immediate once one understands the definitions. The fact that 3 and 4 are equivalent follows from the close relationship between limit points of infinite sets and subsequential limits of sequences.)

An empty set satisfies all five properties vacuously, and hence is compact. In practice, I focus almost exclusively on non-empty compact sets.

2 The Equivalence of Properties 3 and 5.

Theorem 1. *Let (X, d) be a metric space. A non-empty set $C \subseteq X$ is sequentially compact iff it is both complete and totally bounded.*

Proof. $\Rightarrow$. Let C be sequentially compact. Let $\{x_t\}$ be any Cauchy sequence in C. By sequential compactness, $\{x_t\}$ has a subsequence, say x_{t_k}, converging to $x \in C$. Since $\{x_t\}$ is Cauchy, it follows that $\{x_t\}$ converges. Thus C is complete. As for total boundedness, I argue by contraposition. Suppose that there is a $\varepsilon > 0$ such that C is not totally bounded for ε. Choose x_1 to be any point in C. Choose x_2 to be any point in C that is not in $N_\varepsilon(x_1)$. This is possible since C is not totally bounded. Choose x_3 to be any point in C that is not in $N_\varepsilon(x_1) \cup N_\varepsilon(x_2)$. Again, this is possible since C is not totally bounded. And so on. Proceeding in this way, I construct a sequence $\{x_t\}$ such that no term in the sequence is within ε of any other term in the sequence. Thus, this sequence has no Cauchy subsequences, and hence no convergent subsequences, hence is not sequentially compact.

$\Leftarrow$. Let C be complete and totally bounded. Let $\{x_t\}$ be any sequence in C. Let $E \subseteq C$ be the range of $\{x_t\}$. If E is finite then one of the elements of E must be repeated infinitely often, in which case $\{x_t\}$ has a trivial convergent subsequence. Suppose, then, that E is infinite. Form a cover of C using a finite set of balls of radius 1. This is possible since C is totally bounded. One of these balls, label it N_1, must contain an infinite subset of E. Take the first term in the subsequence, x_{t_1}, to be any term in the sequence that lies in $N_1 \cap E$. Now form a new cover of C using a finite set of balls of radius 1/2. One of these balls, label it N_2, must contain an infinite subset of $N_1 \cap E$. Take the second term in the subsequence, x_{t_2}, to be any term in the sequence that lies in $N_2 \cap N_1 \cap E$ and for which $t_2 > t_1$. Continuing in this way, I construct a Cauchy subsequence $\{x_{t_k}\}$. Since C is complete, $\{x_{t_k}\}$ converges to a point of C. Thus C is sequentially compact. ■

In the next set of notes, I show that (a) $\mathbb{R}^N$ is complete and (b) bounded sets in $\mathbb{R}^N$ are totally bounded. It follows that a set in $\mathbb{R}^N$ is (sequentially) compact iff it is closed and bounded. Thus $[0, 1]$ is compact while $[0, 1)$ and $[0, \infty)$ are not. For $[0, 1)$, consider $\{1/2, 3/4/, 7/8, \dots\}$. This converges to 1 but $1 \notin [0, 1)$. For $[0, \infty)$, consider $\{1, 2, 3, \dots\}$. This sequence has no Cauchy subsequences (since all terms are at least 1 apart) and so no subsequence converges.

3 Some Other Properties of Compact Sets.

Even in $\mathbb{R}^N$, a closed set need not be a compact ($[0, \infty)$ is not compact). But, in any metric space, a closed subset of a *compact* set is compact.

Theorem 2. *Let (X, d) be a metric space. Let $C \subseteq X$ be non-empty and compact. If $A \subseteq C$ is closed then A is compact.*

Proof. If C is compact then it is complete and totally bounded. Since $A \subseteq C$ and A is closed, A is likewise complete and it is totally bounded (any subset of a totally bounded set is totally bounded). Therefore A is compact. ■

As one application of compactness, consider the following.

Theorem 3. *Let (X, d) be a metric space. Let $C \subseteq X$ be non-empty and compact and let $\{x_t\}$ be a sequence in C. If every convergent subsequence converges to the same point, x^*, then the sequence converges to x^*.*

Proof. By contraposition. Suppose that C is compact, that $\{x_t\}$ is a sequence in C, but that $x_t \not\to x^*$. Then there is an $\varepsilon > 0$ such that infinitely many $x_t \notin N_\varepsilon(x^*)$, hence there is a subsequence $\{x_{t_k}\}$ for which no term is in $N_\varepsilon(x^*)$. By compactness, $\{x_{t_k}\}$ has a convergent subsequence, which is a subsequence of the original $\{x_t\}$, and by construction the limit of this subsequence cannot be in $N_\varepsilon(x^*)$. Hence there is a convergent subsequence that does not converge to x^*. ■

Note that the hypothesis is weak in that it assumes only that every *convergent* subsequence converges to x^*, not that *every* subsequence converges to x^*. If the latter were true then we would immediately have convergence to x^*. As an example of what can go wrong, suppose that $\{x_t\} = \{1, 2, 1, 3, 1, 4, 1, 5, \dots\}$. Every convergent subsequence here converges to 1 but there are also non-convergent subsequences, such as $\{1, 2, 3, \dots\}$. This $\{x_t\}$ does not lie in a compact set.